Homework assignment #8

Problem 1 (2 pts). A *Gaussian integer* is a complex number of the form $m + in$, where $m, n \in \mathbb{Z}$.

(i) Show that the Gaussian integers form an integral domain.

(ii) Which Gaussian integers have a multiplicative inverse?

(iii) Describe the field of quotients of the Gaussian integers in $\mathbb{C}$.

Problem 2. Let D be an integral domain. Prove that 1 and -1 are the only elements of D that are their own multiplicative inverse.

Problem 3. Let $\mathbb{F}$ be a finite field. Prove that the product of all nonzero elements of $\mathbb{F}$ equals -1. [Hint: use the previous problem.]

Problem 4. Prove that for any prime number p, the number $(p-1)! + 1$ is divisible by p. [Hint: use the previous problem.]

Problem 5 (2 pts). For any integer $n \geq 2$, determine the remainder after division of the number $(n-1)!$ by n.

Problem 6. Determine the remainder after division of the number $2022^{2023^{2024}}$ by 11.

Problem 7. Find all integer solutions of the equation $15x - 23y = 40$.

Problem 8. Find all integer solutions of the equation $13x + 31y = 3$.