Lecture 8:
Subspaces and linear transformations.
Basis and coordinates.
Matrix of a linear transformation.
Linear transformation

Definition. Given vector spaces $V_1$ and $V_2$ over a field $\mathbb{F}$, a mapping $L : V_1 \rightarrow V_2$ is linear if

$$L(x + y) = L(x) + L(y),$$
$$L(rx) = rL(x)$$

for any $x, y \in V_1$ and $r \in \mathbb{F}$.

Basic properties of linear mappings:

- $L(r_1v_1 + \cdots + r_kv_k) = r_1L(v_1) + \cdots + r_kL(v_k)$ for all $k \geq 1$, $v_1, \ldots, v_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{F}$.
- $L(0_1) = 0_2$, where $0_1$ and $0_2$ are zero vectors in $V_1$ and $V_2$, respectively.
- $L(-v) = -L(v)$ for any $v \in V_1$. 
Subspaces and linear maps

Let $V_1, V_2$ be vector spaces and $L : V_1 \to V_2$ be a linear map. Given a set $U \subset V_1$, its image under the map $L$, denoted $L(U)$, is the set of all vectors in $V_2$ that can be represented as $L(x)$ for some $x \in U$.

**Theorem** If $U$ is a subspace of $V_1$ then $L(U)$ is a subspace of $V_2$.

**Proof:** $U$ is nonempty $\implies L(U)$ is nonempty. Let $u, v \in L(U)$. This means $u = L(x)$ and $v = L(y)$ for some $x, y \in U$. By linearity, $u + v = L(x) + L(y) = L(x + y)$. Since $U$ is a subspace of $V_1$, we have $x + y \in U$ so that $u + v \in L(U)$.

Similarly, if $u = L(x)$ for some $x \in U$ then for any $r \in \mathbb{F}$ we have $ru = rL(x) = L(rx) \in L(U)$. 

Subspaces and linear maps

Let \( V_1, V_2 \) be vector spaces and \( L : V_1 \to V_2 \) be a linear map. Given a set \( W \subset V_2 \), its **preimage** (or **inverse image**) under the map \( L \), denoted \( L^{-1}(W) \), is the set of vectors \( x \in V_1 \) such that \( L(x) \in W \).

**Theorem**  If \( W \) is a subspace of \( V_2 \) then its preimage \( L^{-1}(W) \) is a subspace of \( V_1 \).

**Proof:**  Let \( 0_1 \) be the zero vector in \( V_1 \) and \( 0_2 \) be the zero vector in \( V_2 \). By linearity, \( L(0_1) = 0_2 \). Since \( W \) is a subspace of \( V_2 \), it contains \( 0_2 \). Hence \( 0_1 \in L^{-1}(W) \).

Let \( x, y \in L^{-1}(W) \). This means that \( L(x), L(y) \in W \). Then \( L(x + y) = L(x) + L(y) \) is in \( W \) since \( W \) is closed under addition. Therefore \( x + y \in L^{-1}(W) \).

Similarly, if \( L(x) \in W \) for some \( x \in V_1 \) then for any \( r \in \mathbb{F} \) we have \( L(rx) = rL(x) \in W \) so that \( rx \in L^{-1}(W) \).
Range and null-space

Let $V, W$ be vector spaces and $L : V \to W$ be a linear mapping.

**Definition.** The range (or image) of $L$ is the set of all vectors $w \in W$ such that $w = L(v)$ for some $v \in V$. The range of $L$ is denoted $\mathcal{R}(L)$.

The null-space (or kernel) of $L$, denoted $\mathcal{N}(L)$, is the set of all vectors $v \in V$ such that $L(v) = 0$.

**Theorem (i)** The range $\mathcal{R}(L)$ is a subspace of $W$.

**Theorem (ii)** The null-space $\mathcal{N}(L)$ is a subspace of $V$.

**Proof:** $\mathcal{R}(L) = L(V), \mathcal{N}(L) = L^{-1}(\{0\})$. 
Dimension Theorem

**Theorem**  Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space $V$ to a vector space $W$. Then $\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V$.

The null-space $\mathcal{N}(L)$ is a subspace of $V$. It is finite-dimensional since the vector space $V$ is.

Take a basis $v_1, v_2, \ldots, v_k$ for the subspace $\mathcal{N}(L)$, then extend it to a basis $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m$ for the entire space $V$.

**Claim**  Vectors $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Assuming the claim is proved, we obtain

$$\dim \mathcal{R}(L) = m, \quad \dim \mathcal{N}(L) = k, \quad \dim V = k + m.$$
Claim  Vectors $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Proof (spanning): Any vector $w \in \mathcal{R}(L)$ is represented as $w = L(v)$, where $v \in V$. Then

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m$$

for some $\alpha_i, \beta_j \in \mathbb{F}$. By linearity of $L$,

$$w = L(v) = \alpha_1 L(v_1) + \cdots + \alpha_k L(v_k) + \beta_1 L(u_1) + \cdots + \beta_m L(u_m)$$

$$= \beta_1 L(u_1) + \cdots + \beta_m L(u_m).$$

Note that $L(v_i) = 0$ since $v_i \in \mathcal{N}(L)$. Thus $\mathcal{R}(L)$ is spanned by the vectors $L(u_1), \ldots, L(u_m)$. 
Claim  Vectors  $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Proof (linear independence):  Assume that
\[
t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0
\]
for some $t_i \in \mathbb{F}$. Let $u = t_1 u_1 + t_2 u_2 + \cdots + t_m u_m$. Since
\[
L(u) = t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0,
\]
the vector $u$ belongs to the null-space of $L$. Therefore $u = s_1 v_1 + s_2 v_2 + \cdots + s_k v_k$ for some $s_j \in \mathbb{F}$. It follows that
\[
t_1 u_1 + t_2 u_2 + \cdots + t_m u_m - s_1 v_1 - s_2 v_2 - \cdots - s_k v_k = u - u = 0.
\]
Linear independence of vectors $v_1, \ldots, v_k, u_1, \ldots, u_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(u_1), L(u_2), \ldots, L(u_m)$ are linearly independent.
Let $V_1, V_2$ be vector spaces and $L : V_1 \to V_2$ be a linear map.

**Definition.** The map $L$ is **one-to-one** if it maps different vectors from $V_1$ to different vectors in $V_2$. That is, for any $x, y \in V_1$ we have that $x \neq y$ implies $L(x) \neq L(y)$.

The map $L$ is **onto** if any element $y \in V_2$ is represented as $L(x)$ for some $x \in V_1$. If the map $L$ is both one-to-one and onto, then the inverse map $L^{-1} : V_2 \to V_1$ is well defined.

**Theorem** A linear map $L$ is one-to-one if and only if the nullspace $\mathcal{N}(L)$ is trivial.

**Proof:** Let $0_1$ be the zero vector in $V_1$ and $0_2$ be the zero vector in $V_2$. If a vector $x \neq 0_1$ belongs to $\mathcal{N}(L)$, then $L(x) = 0_2 = L(0_1)$ so that $L$ is not one-to-one.

Conversely, assume that $\mathcal{N}(L)$ is trivial. By linearity, $L(x - y) = L(x) - L(y)$ for all $x, y \in V_1$. Therefore $L(x) = L(y) \implies x - y \in \mathcal{N}(L) \implies x = y$. Thus $L$ is one-to-one.
Basis and coordinates

If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for a vector space \( V \), then any vector \( v \in V \) has a unique representation

\[
v = x_1v_1 + x_2v_2 + \cdots + x_nv_n,
\]

where \( x_i \in \mathbb{F} \). The coefficients \( x_1, x_2, \ldots, x_n \) are called the coordinates of \( v \) with respect to the ordered basis \( v_1, v_2, \ldots, v_n \).

The coordinate mapping

vector \( v \)  \mapsto its coordinates \( (x_1, x_2, \ldots, x_n) \)

establishes a one-to-one correspondence between \( V \) and \( \mathbb{F}^n \). This correspondence is linear.
Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) be elements of a vector space \( V \). Define a map \( f : \mathbb{F}^n \to V \) by
\[
f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n.
\]

**Theorem (i)** The map \( f \) is linear.

(ii) If vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are linearly independent then \( f \) is one-to-one.

(iii) If vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) span \( V \) then \( f \) is onto.

(iv) If vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) form a basis for \( V \) then \( f \) is one-to-one and onto.

**Proof:** The map \( f \) is linear since
\[
(x_1 + y_1) \mathbf{v}_1 + (x_2 + y_2) \mathbf{v}_2 + \cdots + (x_n + y_n) \mathbf{v}_n
= (x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n) + (y_1 \mathbf{v}_1 + y_2 \mathbf{v}_2 + \cdots + y_n \mathbf{v}_n),
\]
\[
(rx_1) \mathbf{v}_1 + (rx_2) \mathbf{v}_2 + \cdots + (rx_n) \mathbf{v}_n = r(x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n)
\]
for all \( x_i, y_i, r \in \mathbb{F} \). Further, linear independence of \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) means that the null-space of \( f \) is trivial, which is equivalent to \( f \) being one-to-one. Finally, statement (iii) is obvious while statement (iv) follows from (ii) and (iii).
Examples. • Coordinates of a vector
\( \mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \) relative to the standard basis \( e_1 = (1, 0, \ldots, 0, 0), e_2 = (0, 1, \ldots, 0, 0), \ldots, e_n = (0, 0, \ldots, 0, 1) \) are \( (x_1, x_2, \ldots, x_n) \).

• Coordinates of a matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in M_{2,2}(\mathbb{F})
\] relative to the basis \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) are \( (a, c, b, d) \).

• Coordinates of a polynomial
\( p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n \) relative to the basis \( 1, x, x^2, \ldots, x^n \) are \( (a_0, a_1, \ldots, a_n) \).
Matrix of a linear transformation

Let $V, W$ be vector spaces and $L : V \to W$ be a linear map. Let $\alpha = [v_1, v_2, \ldots, v_n]$ be an ordered basis for $V$ and $\beta = [w_1, w_2, \ldots, w_m]$ be an ordered basis for $W$.

**Definition.** The **matrix** of $L$ relative to the bases $\alpha$ and $\beta$ is an $m \times n$ matrix whose consecutive columns are coordinates of vectors $L(v_1), L(v_2), \ldots, L(v_n)$ relative to the basis $\beta$.

**Notation.** $[w]_\beta$ denotes coordinates of $w$ relative to the ordered basis $\beta$, regarded as a column vector. $[L]_\alpha^\beta$ denotes the matrix of $L$ relative to $\alpha$ and $\beta$. Then

$$[L]_\alpha^\beta = ([L(v_1)]_\beta, [L(v_2)]_\beta, \ldots, [L(v_n)]_\beta).$$
Examples.  

- \( D : \mathcal{P}_2 \rightarrow \mathcal{P}_1, \ (Dp)(x) = p'(x) \).

Let \( \alpha = [1, x, x^2], \ \beta = [1, x] \). Columns of the matrix \([D]_\alpha^\beta\) are coordinates of polynomials \( D1, Dx, Dx^2 \) w.r.t. the basis \( 1, x \).

\[
\begin{align*}
D1 &= 0, \ Dx = 1, \ Dx^2 = 2x \quad \implies \quad [D]_\alpha^\beta = \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\end{align*}
\]

- \( L : \mathcal{P}_2 \rightarrow \mathcal{P}_2, \ (Lp)(x) = p(x + 1) \).

Let us find the matrix \([L]_\alpha^\alpha\):

\[
L1 = 1, \ Lx = 1 + x, \ Lx^2 = (x + 1)^2 = 1 + 2x + x^2.
\]

\[
\implies \ [L]_\alpha^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\]