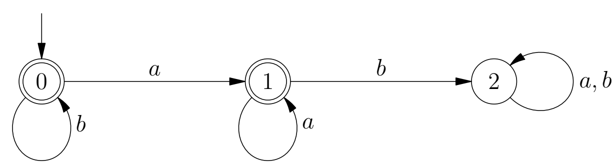
Sample problems for Exam 2

Any problem may be altered, removed or replaced by a different one!

Problem 1. Let R be a relation defined on the set of positive integers by xRy if and only if $\gcd(x, y) \neq 1$ ("is not coprime with"). Is this relation reflexive? Symmetric? Transitive?

Problem 2. A Moore diagram below depicts a 3-state acceptor automaton over the alphabet $\{a, b\}$ which accepts those input words that do not contain a subword ab (and rejects any input word containing a subword ab). Prove that no 2-state automaton can perform the same task.



Problem 3. List all cycles of length 3 in the symmetric group $S(4)$. Make sure there are no repetitions in your list.

Problem 4. Write the permutation $\pi = (4\ 5\ 6)(3\ 4\ 5)(1\ 2\ 3)$ as a product of disjoint cycles.

Problem 5. Find the order and the sign of the permutation $\sigma = (1\,2)(3\,4\,5\,6)(1\,2\,3\,4)(5\,6)$.

Problem 6. What is the largest possible order of an element of the alternating group $A(10)$?

Problem 7. Consider the operation $*$ defined on the set $\mathbb{Z}$ of integers by $a * b = a + b - 2$. Does this operation provide the integers with a group structure?

Problem 8. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

Problem 9. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \quad B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}, \quad C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \quad D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

Problem 10. For any $\lambda \in \mathbb{Q}$ and any $v \in \mathbb{Z}$ let $\lambda \odot v = \lambda v$ if λv is an integer and $\lambda \odot v = v$ otherwise. Does this "selective scaling" make the additive Abelian group $\mathbb{Z}$ into a vector space over the field $\mathbb{Q}$?