Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 The number 63000 has how many positive divisors?

Problem 2 Solve a system of congruences (find all solutions):

$$\begin{cases} x \equiv 2 \bmod 5, \\ x \equiv 3 \bmod 6, \\ x \equiv 6 \bmod 7. \end{cases}$$

Problem 3 Find all integer solutions of a system

$$\begin{cases} 2x + 5y - z = 1, \\ x - 2y + 3z = 2. \end{cases}$$

[Hint: Eliminate one of the variables.]

Problem 4 You receive a message that was encrypted using the RSA system with public key $(55, 27)$, where 55 is the base and 27 is the exponent. The encrypted message, in two blocks, is 4/7. Find the private key and decrypt the message.

Problem 5 Consider a relation $\sim$ on the symmetric group $S(n)$ defined as follows. For any $\pi, \sigma \in S(n)$ we let $\pi \sim \sigma$ if and only if π is conjugate to σ, which means that $\pi = \tau\sigma\tau^{-1}$ for some permutation $\tau \in S(n)$. Show that $\sim$ is an equivalence relation.

Problem 6 Let $\pi = (1\,2)(2\,3)(3\,4)(4\,5)(5\,6)$, $\sigma = (1\,2\,3)(2\,3\,4)(3\,4\,5)(4\,5\,6)$. Find the order and the sign of the following permutations: π, σ, $\pi\sigma$, and $\sigma\pi$.

Problem 7 For any positive integer n let $n\mathbb{Z}$ denote the set of all integers divisible by n. Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under addition? Does it form a group? Explain.

Problem 8 Given a group G, an element $g \in G$ is called central if it commutes with any element of G. The set of all central elements, denoted $C(G)$, is called the center of G. Prove that $C(G)$ is a normal subgroup of G.

Problem 9 (i) List all cyclic subgroups of the alternating group $A(4)$.
(ii) List all non-cyclic subgroups of $A(4)$.

Problem 10 All Abelian groups of order 36 form how many isomorphism classes?

Problem 11 A linear binary coding function f is defined by a generator matrix

$$G = \begin{pmatrix} 0 & \square & 0 & 1 & 1 & 0 & 1 \\ 1 & \square & 0 & 1 & 1 & 1 & 0 \\ 0 & \square & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

with some entries missing. Fill in the missing entries so that f can detect as many errors as possible. Explain.

Problem 12 The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

[Hint: Multiple roots of f are also roots of the derivative f'.]