MATH 433

Applied Algebra

Lecture 23:
Rings and fields.

Groups

Definition. A **group** is a set G, together with a binary operation $*$, that satisfies the following axioms:

(G1: closure)
for all elements g and h of G, $g * h$ is an element of G;

(G2: associativity)
$(g * h) * k = g * (h * k)$ for all $g, h, k \in G$;

(G3: existence of identity)
there exists an element $e \in G$, called the **identity** (or **unit**) of G, such that $e * g = g * e = g$ for all $g \in G$;

(G4: existence of inverse)
for every $g \in G$ there exists an element $h \in G$, called the **inverse** of g, such that $g * h = h * g = e$.

The group $(G, *)$ is said to be **commutative** (or **Abelian**) if it satisfies an additional axiom:

(G5: commutativity) $g * h = h * g$ for all $g, h \in G$.

Semigroups

Definition. A **semigroup** is a nonempty set S, together with a binary operation $*$, that satisfies the following axioms:

(S1: closure)
for all elements g and h of S, $g * h$ is an element of S;

(S2: associativity)
$(g * h) * k = g * (h * k)$ for all $g, h, k \in S$.

The semigroup $(S, *)$ is said to be a **monoid** if it satisfies an additional axiom:

(S3: existence of identity) there exists an element $e \in S$ such that $e * g = g * e = g$ for all $g \in S$.

Optional useful properties of semigroups:

(S4: cancellation) $g * h_1 = g * h_2$ implies $h_1 = h_2$ and $h_1 * g = h_2 * g$ implies $h_1 = h_2$ for all $g, h_1, h_2 \in S$.

(S5: commutativity) $g * h = h * g$ for all $g, h \in S$.

Rings

Definition. A **ring** is a set R, together with two binary operations usually called **addition** and **multiplication** and denoted accordingly, such that

- R is an Abelian group under addition,
- R is a semigroup under multiplication,
- multiplication distributes over addition.

The complete list of axioms is as follows:

(R1) for all $x, y \in R$, $x + y$ is an element of R;

(R2) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$;

(R3) there exists an element, denoted 0, in R such that $x + 0 = 0 + x = x$ for all $x \in R$;

(R4) for every $x \in R$ there exists an element, denoted $-x$, in R such that $x + (-x) = (-x) + x = 0$;

(R5) $x + y = y + x$ for all $x, y \in R$;

(R6) for all $x, y \in R$, xy is an element of R;

(R7) $(xy)z = x(yz)$ for all $x, y, z \in R$;

(R8) $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$ for all $x, y, z \in R$.

Examples of rings

Informally, a ring is a set with three arithmetic operations: addition, subtraction and multiplication. Subtraction is defined by $x - y = x + (-y)$.

- Real numbers $\mathbb{R}$.
- Integers $\mathbb{Z}$.
- $2\mathbb{Z}$: even integers.
- $\mathbb{Z}_n$: congruence classes modulo n.
- $\mathcal{M}_n(\mathbb{R})$: all $n \times n$ matrices with real entries.
- $\mathcal{M}_n(\mathbb{Z})$: all $n \times n$ matrices with integer entries.
- All functions $f : S \to \mathbb{R}$ on a nonempty set S.
- **Zero ring**: any additive Abelian group with trivial multiplication: $xy = 0$ for all x and y.
- Trivial ring $\{0\}$.

Examples of rings

In examples below, real numbers $\mathbb{R}$ can be replaced by a more general ring of coefficients.

- $\mathbb{R}[X]$: polynomials in variable X with real coefficients.

$p(X) = c_0 + c_1X + c_2X^2 + \cdots + c_nX^n$, where each $c_i \in \mathbb{R}$.

- $\mathbb{R}(X)$: rational functions in variable X with real coefficients.

$r(X) = \frac{a_0+a_1X+a_2X^2+\cdots+a_nX^n}{b_0+b_1X+b_2X^2+\cdots+b_mX^m}$, where $a_i, b_j \in \mathbb{R}$ and $b_m \neq 0$.

- $\mathbb{R}[X, Y]$: polynomials in variables X, Y with real coefficients.

$\mathbb{R}[X, Y] = \mathbb{R}[X][Y]$.

- $\mathbb{R}[[X]]$: formal power series in variable X with real coefficients.

$p(X) = c_0 + c_1X + c_2X^2 + \cdots + c_nX^n + \ldots$, where $c_i \in \mathbb{R}$.
Multiplication is well defined. For example,

$$(1 - X)(1 + X + X^2 + X^3 + X^4 + \ldots) = 1.$$

Example. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where $x, y \in \mathbb{R}$.

$$\begin{aligned}
\begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} x' & -y' \\ y' & x' \end{pmatrix} &= \begin{pmatrix} x + x' & -(y + y') \\ y + y' & x + x' \end{pmatrix}, \\
-\begin{pmatrix} x & -y \\ y & x \end{pmatrix} &= \begin{pmatrix} -x & -(-y) \\ -y & -x \end{pmatrix}, \\
\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x' & -y' \\ y' & x' \end{pmatrix} &= \begin{pmatrix} xx' - yy' & -(xy' + yx') \\ xy' + yx' & xx' - yy' \end{pmatrix}.
\end{aligned}$$

Hence M is closed under matrix addition, taking the negative, and matrix multiplication. Also, the multiplication is commutative on M. The associativity and commutativity of the addition, the associativity of the multiplication, and the distributive law hold on M since they hold for all 2×2 matrices. Thus M is a commutative ring.

Remark. M is the ring of complex numbers $x + yi$ "in disguise".

Divisors of zero

Theorem Let R be a ring. Then $x0 = 0x = 0$ for all $x \in R$.

Proof: Let $y = x0$. Then $y + y = x0 + x0 = x(0 + 0)$ $= x0 = y$. It follows that $(-y) + y + y = (-y) + y$, hence $y = 0$. Similarly, one shows that $0x = 0$.

A nonzero element x of a ring R is a **left zero-divisor** if $xy = 0$ for another nonzero element $y \in R$. The element y is called a **right zero-divisor**.

Examples. • In the ring $\mathbb{Z}_6$, the zero-divisors are congruence classes $[2]_6$, $[3]_6$, and $[4]_6$, as $[2]_6[3]_6 = [4]_6[3]_6 = [0]_6$.

• In the ring $\mathcal{M}_n(\mathbb{R})$, the zero-divisors (both left and right) are nonzero matrices with zero determinant. For instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

• In any zero ring, all nonzero elements are zero-divisors.

Integral domains

A ring R is called a **domain** if it has no zero-divisors.

Theorem Given a nontrivial ring R, the following are equivalent:

- R is a domain,
- $R \setminus \{0\}$ is a semigroup under multiplication,
- $R \setminus \{0\}$ is a semigroup with cancellation under multiplication.

Idea of the proof: No zero-divisors means that $R \setminus \{0\}$ is closed under multiplication. Further, if $a \neq 0$ then $ab = ac$ $\implies a(b-c) = 0 \implies b-c = 0 \implies b = c$.

A ring R is called **commutative** if the multiplication is commutative. R is called a **ring with identity** if there exists an identity element for multiplication (denoted 1).

An **integral domain** is a nontrivial commutative ring with identity and no zero-divisors.

Fields

Definition. A **field** is a set F, together with two binary operations called **addition** and **multiplication** and denoted accordingly, such that

- F is an Abelian group under addition,
- $F \setminus \{0\}$ is an Abelian group under multiplication,
- multiplication distributes over addition.

In other words, the field is a commutative ring with identity $(1 \neq 0)$ such that any nonzero element has a multiplicative inverse.

Examples. • Real numbers $\mathbb{R}$.

- Rational numbers $\mathbb{Q}$.
- Complex numbers $\mathbb{C}$.
- $\mathbb{Z}_p$: congruence classes modulo p, where p is prime.
- $\mathbb{R}(X)$: rational functions in variable X with real coefficients.

From rings to fields

Theorem Any finite integral domain is, in fact, a field.

Theorem A ring R with identity can be extended to a field if and only if it is an integral domain.

If R is an integral domain, then there is a smallest field F containing R called the **quotient field** of R. Any element of F is of the form $b^{-1}a$, where $a, b \in R$.

Examples. • The quotient field of $\mathbb{Z}$ is $\mathbb{Q}$.
• The quotient field of $\mathbb{R}[X]$ is $\mathbb{R}(X)$.