MATH 433

Applied Algebra

Lecture 17:
Permutations (continued).
Cycle decomposition.

Permutations

Let X be a finite set. A **permutation** of X is a bijection from X to itself. Permutations are traditionally denoted by Greek letters (π, σ, τ, ρ,...).

Two-row notation. $\pi = \begin{pmatrix} a & b & c & \dots \\ \pi(a) & \pi(b) & \pi(c) & \dots \end{pmatrix}$,

where $a, b, c, \dots$ is a list of all elements in the domain of π.

The set of all permutations of a finite set X is called the **symmetric group** on X. *Notation:* S_X, Σ_X, $\mathrm{Sym}(X)$.
The set of all permutations of $\{1, 2, \dots, n\}$ is called the **symmetric group** on n symbols and denoted $S(n)$ or S_n.

Given two permutations π and σ, the composition $\pi\sigma$, defined by $\pi\sigma(x) = \pi(\sigma(x))$, is called the **product** of these permutations. In general, $\pi\sigma \neq \sigma\pi$, i.e., multiplication of permutations is not commutative. However, it is associative: $\pi(\sigma\tau) = (\pi\sigma)\tau$.

Example. The symmetric group $S(3)$ consists of 6 permutations:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}.$$

Theorem The symmetric group $S(n)$ has $n! = 1 \cdot 2 \cdot 3 \cdots n$ elements.

Traditional argument: The number of elements in $S(n)$ is the number of different rearrangements $x_1, x_2, \dots, x_n$ of the list $1, 2, \dots, n$. There are n possibilities to choose x_1. For any choice of x_1, there are $n-1$ possibilities to choose x_2. And so on...

Alternative argument: Any rearrangement of the list $1, 2, \dots, n$ can be obtained as follows. We take a rearrangement of $1, 2, \dots, n-1$ and then insert n into it. By the inductive assumption, there are $(n-1)!$ ways to choose a rearrangement of $1, 2, \dots, n-1$. For any choice, there are n ways to insert n.

Product of permutations

Let π and σ be two permutations of the same set. To find the product $\pi\sigma$, we write π underneath σ (in two-row notation), then reorder the columns so that the second row of σ matches the first row of π, then erase the matching rows.

Example. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$.

$$\begin{matrix} \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} \\ \pi = \begin{pmatrix} 3 & 2 & 1 & 5 & 4 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix} \end{matrix} \quad \Longrightarrow \quad \pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$$

To find π^{-1}, we simply exchange the upper and lower rows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Cycles

A permutation π of a set X is called a **cycle** (or **cyclic**) of length r if there exist r distinct elements $x_1, x_2, \ldots, x_r \in X$ such that

$$\pi(x_1) = x_2,\ \pi(x_2) = x_3, \ldots,\ \pi(x_{r-1}) = x_r,\ \pi(x_r) = x_1,$$

and $\pi(x) = x$ for any other $x \in X$.

Notation. $\pi = (x_1\ x_2\ \ldots\ x_n)$.

The identity function is (the only) cycle of length 1. Any cycle of length 2 is called a **transposition**.

The inverse of a cycle is also a cycle of the same length. Indeed, if $\pi = (x_1\ x_2\ \ldots\ x_n)$, then $\pi^{-1} = (x_n\ x_{n-1}\ \ldots\ x_2\ x_1)$.

Example. Any permutation of $\{1, 2, 3\}$ is a cycle.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \mathrm{id},\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2\ 3),\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2),$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3),\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2),\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3).$$

Cycle decomposition

Let π be a permutation of X. We say that π **moves** an element $x \in X$ if $\pi(x) \neq x$. Otherwise π **fixes** x.

Two permutations π and σ are called **disjoint** if the set of elements moved by π is disjoint from the set of elements moved by σ.

Theorem If π and σ are disjoint permutations in S_X, then they commute: $\pi\sigma = \sigma\pi$.

Idea of the proof: If π moves an element x, then it also moves $\pi(x)$. Hence σ fixes both so that $\pi\sigma(x) = \sigma\pi(x) = \pi(x)$.

Theorem Any permutation of a finite set can be expressed as a product of disjoint cycles. This **cycle decomposition** is unique up to rearrangement of the cycles involved.

Idea of the proof: Given $\pi \in S_X$, for any $x \in X$ consider a sequence $a_1 = x, a_2, a_3, \dots,$ where $a_{m+1} = \pi(a_m)$. Let r be the least index such that $a_r = a_k$ for some $k < r$. Then $k = 1$.