MATH 433

Applied Algebra

Lecture 22:
Basic properties of groups.
Cayley table.
Transformation groups.

Abstract groups

Definition. A **group** is a set G, together with a binary operation $*$, that satisfies the following axioms:

(G1: closure)
for all elements g and h of G, $g * h$ is an element of G;

(G2: associativity)
$(g * h) * k = g * (h * k)$ for all $g, h, k \in G$;

(G3: existence of identity)
there exists an element $e \in G$, called the **identity** (or **unit**) of G, such that $e * g = g * e = g$ for all $g \in G$;

(G4: existence of inverse)
for every $g \in G$ there exists an element $h \in G$, called the **inverse** of g, such that $g * h = h * g = e$.

The group $(G, *)$ is said to be **commutative** (or **Abelian**) if it satisfies an additional axiom:

(G5: commutativity) $g * h = h * g$ for all $g, h \in G$.

Basic properties of groups

- The identity element is unique.
- The inverse element is unique.
- $(ab)^{-1} = b^{-1}a^{-1}$.
- $(a_1 a_2 \dots a_n)^{-1} = a_n^{-1} \dots a_2^{-1} a_1^{-1}$.

Basic properties of groups

- **Cancellation properties**: $ab = ac \implies b = c$ and $ba = ca \implies b = c$ for all $a, b, c \in G$.

Indeed, $ab = ac \implies a^{-1}(ab) = a^{-1}(ac)$
$\implies (a^{-1}a)b = (a^{-1}a)c \implies eb = ec \implies b = c$.
Similarly, $ba = ca \implies (ba)a^{-1} = (ca)a^{-1}$
$\implies b(aa^{-1}) = c(aa^{-1}) \implies be = ce \implies b = c$.

- If $hg = g$ or $gh = g$ for some $g \in G$, then h is the identity element.

Indeed, $hg = g \implies hg = eg$. By right cancellation, $h = e$.
Likewise, $gh = g \implies gh = ge$. By left cancellation, $h = e$.

- $gh = e \iff hg = e \iff h = g^{-1}$.

$$gh = e \iff gh = gg^{-1} \iff h = g^{-1} \iff hg = g^{-1}g \iff hg = e$$

Cayley table

A binary operation on a finite set can be given by a **Cayley table** (i.e., "multiplication" table):

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The Cayley table is convenient to check commutativity of the operation (the table should be symmetric relative to the diagonal), cancellation properties (left cancellation holds if each row contains all elements, right cancellation holds if each column contains all elements), existence of the identity element, and existence of the inverse.

However this table is not convenient to check associativity of the operation.

Problem. The following is a partially completed Cayley table for a certain commutative group:

$*$	a	b	c	d
a	b			c
b			c	
c				a
d		d		

Complete the table.

Solution:

$*$	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	b	a
d	c	d	a	b

Transformation groups

Definition. A **transformation group** is a group of bijective transformations of a set X with the operation of composition.

Examples.

- Symmetric group $S(n)$: all permutations of $\{1, 2, \ldots, n\}$.
- Alternating group $A(n)$: even permutations of $\{1, 2, \ldots, n\}$.
- $\mathrm{Homeo}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuous (such functions are called **homeomorphisms**).
- $\mathrm{Homeo}^+(\mathbb{R})$: the group of all increasing functions in $\mathrm{Homeo}(\mathbb{R})$ (i.e., those that preserve orientation of the real line).
- $\mathrm{Diff}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuously differentiable (such functions are called **diffeomorphisms**).

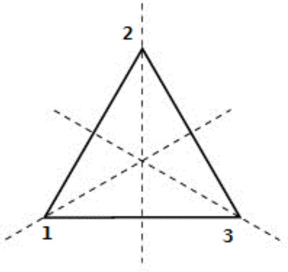
Groups of symmetries

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a **motion** (or a **rigid motion**) if it preserves distances between points.

Theorem All motions of $\mathbb{R}^n$ form a transformation group. Any motion $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix ($A^T A = AA^T = I$).

Given a geometric figure $F \subset \mathbb{R}^n$, a **symmetry** of F is a motion of $\mathbb{R}^n$ that preserves F. All symmetries of F form a transformation group.

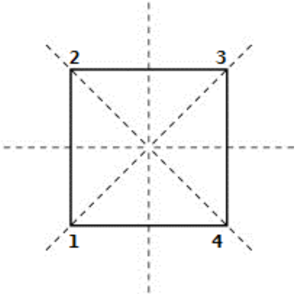
Example. • The **dihedral group** $D(n)$ is the group of symmetries of a regular n-gon. It consists of $2n$ elements: n reflections, $n-1$ rotations by angles $2\pi k/n$, $k = 1, 2, \ldots, n-1$, and the identity function.



Equlateral triangle

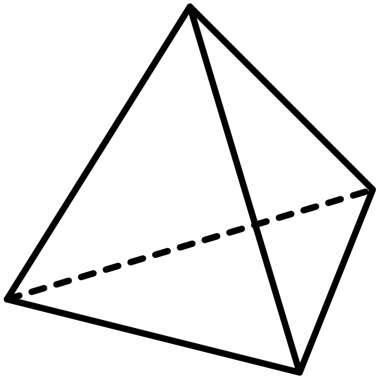
Any symmetry of a polygon maps vertices to vertices. Therefore it induces a permutation on the set of vertices. Moreover, the symmetry is uniquely recovered from the permutation.

In the case of the equilateral triangle, any permutation of vertices comes from a symmetry.



Square

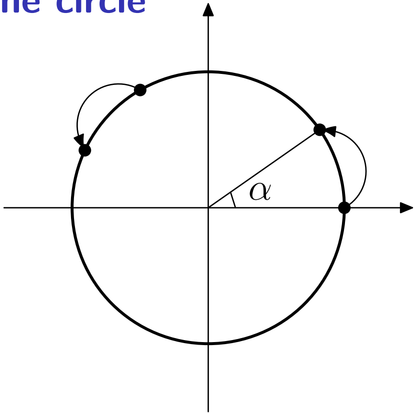
In the case of the square, not every permutation of vertices comes from a symmetry of the square. The reason is that a symmetry must map adjacent vertices to adjacent vertices.



Regular tetrahedron

Any symmetry of a polyhedron maps vertices to vertices. In the case of the regular tetrahedron, any permutation of vertices comes from a symmetry.

Rotations of the circle



Let $R_\alpha : S^1 \to S^1$ be the rotation of the circle S^1 by angle $\alpha \in \mathbb{R}$. All rotations R_α, $\alpha \in \mathbb{R}$ form a transformation group. Namely, $R_\alpha R_\beta = R_{\alpha+\beta}$, $R_\alpha^{-1} = R_{-\alpha}$, and $R_0 = \mathrm{id}$.

The group of rotations is part (a **subgroup**) of the group of all symmetries of the circle (the other symmetries are reflections).

Matrix groups

A group is called **linear** if its elements are $n \times n$ matrices and the group operation is matrix multiplication.

- **General linear group** $GL(n, \mathbb{R})$ consists of all $n \times n$ matrices that are invertible (i.e., with nonzero determinant).
  The identity element is $I = \mathrm{diag}(1, 1, \ldots, 1)$.

- **Special linear group** $SL(n, \mathbb{R})$ consists of all $n \times n$ matrices with determinant 1.
  Closed under multiplication since $\det(AB) = \det(A)\det(B)$. Also, $\det(A^{-1}) = (\det(A))^{-1}$.

- **Orthogonal group** $O(n, \mathbb{R})$ consists of all orthogonal $n \times n$ matrices $(A^T = A^{-1})$.

- **Special orthogonal group** $SO(n, \mathbb{R})$ consists of all orthogonal $n \times n$ matrices with determinant 1.
  $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$.