MATH 614

Dynamical Systems and Chaos

Lecture 3: Classification of fixed points.

Periodic points

Definition. A point $x \in X$ is called a **fixed** point of a map $f : X \to X$ if $f(x) = x$.

A point $x \in X$ is called a **periodic** point of a map $f : X \to X$ if $f^m(x) = x$ for some integer $m \geq 1$. The least integer m satisfying this relation is called the **prime period** of x.

A point $x \in X$ is called an **eventually periodic** point of the map f if for some integer $k \geq 0$ the point $f^k(x)$ is a periodic point of f.

Properties of periodic points

- If x is a periodic point of prime period m, then $f^n(x) = x$ if and only if m divides n.
- If x is a periodic point of prime period m, then $f^{n_1}(x) = f^{n_2}(x)$ if and only if m divides $n_1 - n_2$.
- If x is a periodic point of prime period m, then the orbit of x consists of m points.
- If x is a periodic point, then every element of the orbit of x is also a periodic point of the same period.
- A point is eventually periodic if and only if its orbit is finite (as a set).
- If the map f is invertible, then every eventually periodic point is actually periodic.

Classification of fixed points

Let X be a subset of $\mathbb{R}$, $f : X \to X$ be a continuous map, and x_0 be a fixed point of f.

Definition. The **stable set** of the fixed point x_0, denoted $W^s(x_0)$, consists of all points $x \in X$ such that $f^n(x) \to x_0$ as $n \to \infty$. In the case f is invertible, the **unstable set** of x_0, denoted $W^u(x_0)$, is the stable set of x_0 considered as a fixed point of f^{-1}.

The fixed point x_0 is **weakly attracting** if the stable set $W^s(x_0)$ contains an open neighborhood of x_0, i.e., $(x_0 - \varepsilon, x_0 + \varepsilon) \subset W^s(x_0)$ for some $\varepsilon > 0$.

The fixed point x_0 is **weakly repelling** if there exists an open neighborhood U of x_0 such that for each $x \in U \setminus \{x_0\}$ the orbit $O^+(x)$ is not completely contained in U.

Proposition Suppose f is invertible. Then a fixed point x_0 of f is weakly repelling if and only if x_0 is a weakly attracting fixed point of the inverse map f^{-1}.

Proof: It is no loss to assume that the domain of f includes an interval $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. Since the map f is continuous and invertible, it is strictly monotone on U. Let us choose ε_0, $0 < \varepsilon_0 < \varepsilon$, so that $f(U_0) \subset U$, where $U_0 = (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. Then f^2 is strictly increasing on U_0.

If x_0 is not an isolated fixed point of f^2, then it is neither weakly attracting nor weakly repelling for both f and f^{-1}. Therefore it is no loss to assume that x_0 is the only fixed point of f^2 in the interval U_0. Then the function $g(x) = f^2(x) - x$ maintains its sign on $(x_0 - \varepsilon_0, x_0)$ and on $(x_0, x_0 + \varepsilon_0)$. If those signs are $-$ and $+$, then x_0 is both weakly repelling for f^2 and weakly attracting for f^{-2}. Otherwise x_0 is neither.

Finally, x_0 is weakly repelling for f if and only if it is so for f^2. Also, x_0 is weakly attracting for f^{-1} if and only if it is for f^{-2}.

Classification of fixed points (continued)

Definition. The fixed point x_0 is **attracting** if for some $\lambda \in (0,1)$ there exists an open interval U containing x_0 such that $|f(x) - x_0| \leq \lambda |x - x_0|$ for all $x \in U$. The point x_0 is **super-attracting** if such an interval exists for any $\lambda \in (0,1)$.

It is no loss to assume that $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. Then it follows that $f(U) \subset U$ and $|f^n(x) - x_0| \leq \lambda^n |x - x_0|$ for all $x \in U$ and $n = 1, 2, \dots$ In particular, the orbit of any point $x \in U$ converges to x_0. Hence an attracting fixed point is weakly attracting as well.

Definition. The fixed point x_0 is **repelling** if there exist $\lambda > 1$ and an open interval U containing x_0 such that $|f(x) - x_0| \geq \lambda |x - x_0|$ for all $x \in U$.

It is easy to observe that any repelling fixed point is weakly repelling as well.

Theorem Suppose that a map $f : X \to X$ is differentiable at a fixed point x_0 and let $\lambda = f'(x_0)$ be the multiplier of x_0. Then **(i)** x_0 is attracting if and only if $|\lambda| < 1$;
(ii) x_0 is super-attracting if and only if $\lambda = 0$;
(iii) x_0 is repelling if and only if $|\lambda| > 1$.

Proof: Since $\lambda = f'(x_0)$, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\lambda - \delta < \frac{f(x) - x_0}{x - x_0} < \lambda + \delta \quad \text{whenever} \quad 0 < |x - x_0| < \varepsilon.$$

Then $(|\lambda| - \delta)|x - x_0| \leq |f(x) - x_0| \leq (|\lambda| + \delta)|x - x_0|$ for $|x - x_0| < \varepsilon$. Notice that the numbers $|\lambda| - \delta$ and $|\lambda| + \delta$ can be made arbitrarily close to $|\lambda|$.

In the case $|\lambda| < 1$, we obtain that x_0 is an attracting fixed point. Furthermore, if $\lambda = 0$, then x_0 is super-attracting. However x_0 is not super-attracting if $\lambda \neq 0$. In the case $|\lambda| > 1$, we obtain that x_0 is a repelling fixed point. Finally, in the case $|\lambda| = 1$ the fixed point x_0 is neither attracting nor repelling.

Classification of periodic points

Let X be a subset of $\mathbb{R}$, $f : X \to X$ be a continuous map, and x_0 be a periodic point of f with prime period m. Then x_0 is a fixed point of the map f^m.

The **stable set** of the periodic point x_0, denoted $W^s(x_0)$, is defined as the stable set of the same point considered a fixed point of f^m. That is, $W^s(x_0)$ consists of all points $x \in X$ such that $f^{nm}(x) \to x_0$ as $n \to \infty$.

In the case f is invertible, so is the map f^m. In this case the **unstable set** of x_0, denoted $W^u(x_0)$, is defined as the stable set of x_0 considered a fixed point of $(f^m)^{-1}$.

The periodic point x_0 is called **weakly attracting** (resp. **attracting**, **super-attracting**, **weakly repelling**, **repelling**) if it enjoys the same property as a fixed point of f^m.

Newton's method

Newton's method is an iterative process for finding roots of a polynomial. Given a nonconstant polynomial Q, consider a rational function

$$f(x) = x - \frac{Q(x)}{Q'(x)}.$$

It turns out that, for a properly chosen initial point x_0, the orbit $x_0, f(x_0), f^2(x_0), \dots$ converges very fast to a root of Q.

Suppose z is a simple root of Q, that is, $Q(z) = 0$ while $Q'(z) \neq 0$. Clearly, z is a fixed point of the map f. We have

$$f'(z) = 1 - \frac{Q'(z)Q'(z) - Q(z)Q''(z)}{(Q'(z))^2} = \frac{Q(z)Q''(z)}{(Q'(z))^2} = 0.$$

Thus z is a super-attracting fixed point of f.