MATH 614

Dynamical Systems and Chaos

Lecture 6:
Symbolic dynamics.

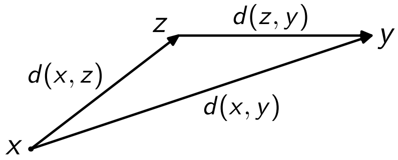
Metric space

Definition. Given a nonempty set X, a **metric** (or **distance function**) on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

• **(positivity)** $d(x, y) \geq 0$ for all $x, y \in X$; moreover, $d(x, y) = 0$ if and only if $x = y$;

• **(symmetry)** $d(x, y) = d(y, x)$ for all $x, y \in X$;

• **(triangle inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.



A set endowed with a metric is called a **metric space**.

Open sets

Let (X, d) be a metric space. For any $x_0 \in X$ and $\varepsilon > 0$ we define the **open ball** (or simply **ball**) $B_\varepsilon(x_0)$ of radius ε centered at x_0 by $B_\varepsilon(x_0) = \{x \in X \mid d(x, x_0) < \varepsilon\}$.
The ball $B_\varepsilon(x_0)$ is also called the ε-**neighborhood** of x_0.

A subset U of the metric space X is called **open** if for every point $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Let (Y, ρ) be another metric space and $f : X \to Y$ be a function.

Proposition 1 The function f is continuous at a point $x \in X$ if and only if for every open set $W \subset Y$ containing $f(x)$ there is an open set $U \subset X$ containing x such that $f(U) \subset W$.

Proposition 2 The function f is continuous on the entire set X if and only if for any open set $W \subset Y$ the preimage $f^{-1}(W)$ is an open set in X.

Topological space

Definition. Given a nonempty set X, a **topology** on X is a collection $\mathcal{U}$ of subsets of X such that

- $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$,
- any intersection of finitely many elements of $\mathcal{U}$ is also in $\mathcal{U}$,
- any union of elements of $\mathcal{U}$ is also in $\mathcal{U}$.

Elements of $\mathcal{U}$ are referred to as **open sets** of the topology. A set endowed with a topology is called a **topological space**.

We say that a sequence of points $x_1, x_2, \dots$ of the topological space X **converges** to a point $y \in X$ if for every open set $U \in \mathcal{U}$ containing y there exists a natural number n_0 such that $x_n \in U$ for $n \geq n_0$.

Given another topological space Y and a function $f : X \to Y$, we say that f is **continuous** if for any open set $W \subset Y$ the preimage $f^{-1}(W)$ is an open set in X.

Examples of topological spaces

- *Metric space*

X: a metric space, $\mathcal{U}$: the set of all open subsets of X ($\mathcal{U}$ is referred to as the topology induced by the metric).

- *Trivial topology*

X: any nonempty set, $\mathcal{U} = \{\emptyset, X\}$.

- *Discrete topology*

X: any nonempty set, $\mathcal{U}$: the set of all subsets of X.

- *Subspace of a topological space*

X: nonempty subset of a topological space Y with a topology $\mathcal{W}$, $\mathcal{U} = \{U \cap X \mid U \in \mathcal{W}\}$.

Space of infinite sequences

Let $\mathcal{A}$ be a finite set. We denote by $\Sigma_{\mathcal{A}}$ the set of all infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$. Elements of $\Sigma_{\mathcal{A}}$ are also referred to as **infinite words** over the **alphabet** $\mathcal{A}$.

For any finite sequence $s_1 s_2 \dots s_n$ of elements of $\mathcal{A}$ let $C(s_1 s_2 \dots s_n)$ denote the set of all infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that begin with this sequence. The sets $C(s_1 s_2 \dots s_n)$ are called **cylinders**. Let $\mathcal{U}$ be the collection of all subsets of $\Sigma_{\mathcal{A}}$ that can be represented as unions of cylinders.

Proposition 1 $\mathcal{U}$ is a topology on $\Sigma_{\mathcal{A}}$.

The topological space $(\Sigma_{\mathcal{A}}, \mathcal{U})$ is **metrizable**, which means that the topology $\mathcal{U}$ is induced by a metric on $\Sigma_{\mathcal{A}}$. For any $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}$ let $d(\mathbf{s}, \mathbf{t}) = 2^{-n}$ if $s_i = t_i$ for $1 \leq i \leq n$ while $s_{n+1} \neq t_{n+1}$. Also, let $d(\mathbf{s}, \mathbf{t}) = 0$ if $s_i = t_i$ for all $i \geq 1$.

Proposition 2 The function d is a metric on $\Sigma_{\mathcal{A}}$ that induces the topology $\mathcal{U}$.

Symbolic dynamics

The symbolic dynamics is concerned with the study of some continuous transformations of the topological space $\Sigma_{\mathcal{A}}$ of infinite words over a finite alphabet $\mathcal{A}$. The most important of them is the **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$.

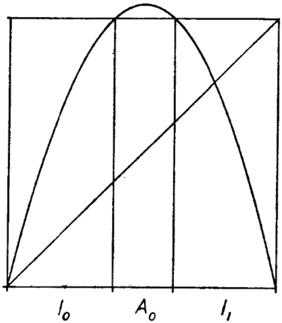
Proposition The shift transformation is continuous.

Proof: We have to show that for any open set $W \subset \Sigma_{\mathcal{A}}$ the preimage $\sigma^{-1}(W)$ is also open. The set W is a union of cylinders: $W = \bigcup_{\beta \in B} C_\beta$. Since

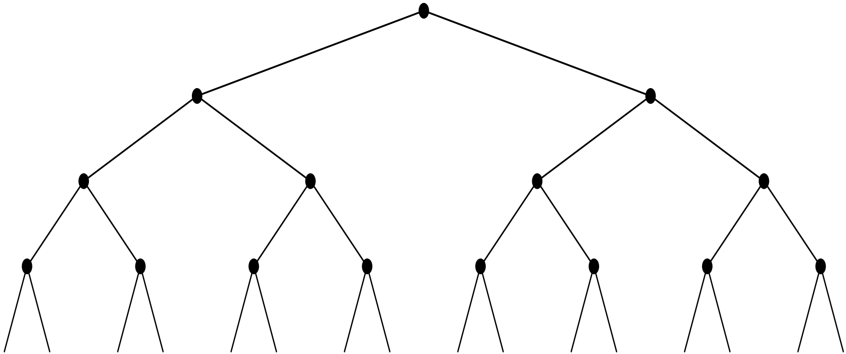
$$\sigma^{-1}\Bigl(\bigcup_{\beta \in B} C_\beta\Bigr) = \bigcup_{\beta \in B} \sigma^{-1}(C_\beta),$$

it is enough to show that the preimage of any cylinder C_β is open. Let $C_\beta = C(s_1 s_2 \dots s_n)$. Then $\sigma^{-1}(C_\beta)$ is the union of cylinders $C(s_0 s_1 s_2 \dots s_n)$, $s_0 \in \mathcal{A}$, hence it is open.

Unimodal maps



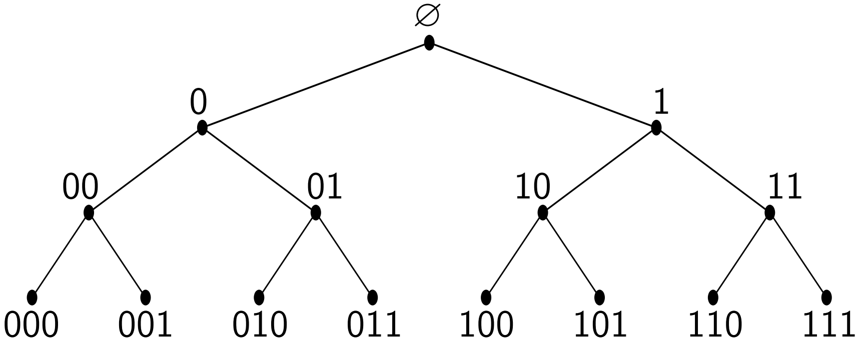
Binary rooted tree



Alphabet $\mathcal{A} = \{0, 1\}$

$\mathcal{A}^*$: the set of words in the alphabet $\mathcal{A}$

Binary rooted tree



Alphabet $\mathcal{A} = \{0, 1\}$

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Continuity of the itinerary map

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_2 = \Sigma_{\{0,1\}}$ be the itinerary map.

Proposition The itinerary map S is continuous.

Proof: Since every open subset of Σ_2 is a union of cylinders, it is enough to show that for any cylinder $C = C(s_0 s_1 \dots s_n)$ the preimage $S^{-1}(C)$ is an open subset of Λ, i.e., $S^{-1}(C) = U \cap \Lambda$, where U is an open subset of $\mathbb{R}$.

Clearly, $S^{-1}(C) = I_{s_0 s_1 \dots s_n} \cap \Lambda$, where

$$I_{s_0 s_1 \dots s_n} = \{x \in [0,1] \mid f^k(x) \in I_{s_k},\ 0 \le k \le n\}.$$

We know that $I_{s_0 s_1 \dots s_n}$ is a closed interval and the set Λ is covered by 2^{n+1} disjoint closed intervals of the form $I_{t_0 t_1 \dots t_n}$, where each t_i is 0 or 1. It follows that there exists an open interval U such that $S^{-1}(C) = I_{s_0 s_1 \dots s_n} \cap \Lambda = U \cap \Lambda$.

Periodic points of the shift

Definition. A point $x \in X$ is a **periodic** point of **period** n of a map $f : X \to X$ if $f^n(x) = x$. The least $n \geq 1$ satisfying this relation is called the **prime period** of x.

Suppose $\mathbf{s} \in \Sigma_{\mathcal{A}}$. Given a natural number n, let $\mathbf{s}' = \sigma^n(\mathbf{s})$ and w be the beginning of length n of $\mathbf{s}$. Then $\mathbf{s} = w\mathbf{s}'$. It follows that $\sigma^n(\mathbf{s}) = \mathbf{s}$ if and only if $\mathbf{s} = www\dots$ Similarly, an infinite word $\mathbf{t}$ is an eventually periodic point of the shift if and only if $\mathbf{t} = uwww\dots$ for some finite words u and w.

Proposition The number of periodic points of period n is k^n, where k is the number of elements in the alphabet $\mathcal{A}$.

Idea of the proof: The number of periodic points of period n equals the number of finite words of length n, which is k^n.

Dense sets

Definition. Suppose (X, d) is a metric space. We say that a subset $E \subset X$ is **everywhere dense** (or simply **dense**) in X if for every $x \in X$ and $\varepsilon > 0$ there exists $y \in E$ such that $d(y, x) < \varepsilon$.

More generally, suppose X is a topological space. We say that a subset $E \subset X$ is **dense** in X if E intersects every nonempty open subset of X.

Proposition Periodic points of the shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ are dense in $\Sigma_{\mathcal{A}}$.

Proof: Let w be any nonempty finite word over the alhabet $\mathcal{A}$. Then the cylinder $C(w)$ contains a periodic point, e.g., $www\dots$ Consequently, any nonempty open set $U \subset \Sigma_{\mathcal{A}}$ contains a periodic point.

Dense orbit of the shift

Proposition The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ admits a dense orbit.

Proof: Since open subsets of $\Sigma_{\mathcal{A}}$ are unions of cylinders, it follows that a set $E \subset \Sigma_{\mathcal{A}}$ is dense if and only if it intersects every cylinder.

The orbit under the shift of an infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ visits a particular cylinder $C(w)$ if and only if the finite word w appears somewhere in $\mathbf{s}$, that is, $\mathbf{s} = w_0 w \mathbf{s}_0$, where w_0 is a finite word and $\mathbf{s}_0$ is an infinite word. Therefore the orbit $O_\sigma^+(\mathbf{s})$ is dense in $\Sigma_{\mathcal{A}}$ if and only if the infinite word $\mathbf{s}$ contains all finite words over the alphabet $\mathcal{A}$ as subwords.

There are only countably many finite words over $\mathcal{A}$. We can enumerate them all: $w_1, w_2, w_3, \dots$ Then an infinite word $\mathbf{s} = w_1 w_2 w_3 \dots$ has dense orbit.