MATH 614

Dynamical Systems and Chaos

Lecture 9:
Compact sets.
Definition of chaos.

Topological conjugacy

Suppose $f : X \to X$ and $g : Y \to Y$ are transformations of topological spaces.

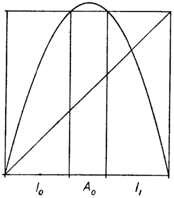
Definition. We say that a map $\phi : X \to Y$ is a **semi-conjugacy** of f with g if ϕ is onto and $\phi \circ f = g \circ \phi$.

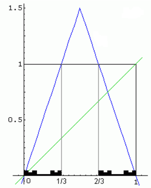
$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \phi\Big\downarrow & & \Big\downarrow\phi \\ Y & \xrightarrow{g} & Y \end{array}$$

The map ϕ is a **conjugacy** if, additionally, it is invertible. The map ϕ is a **topological conjugacy** if, additionally, it is a homeomorphism, which means that both ϕ and ϕ^{-1} are continuous. In the latter case, we say that the maps f and g are **topologically conjugate**. Note that $f = \phi^{-1} g \phi$ and $g = \phi f \phi^{-1}$.

Unimodal maps

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_2 = \Sigma_{\{0,1\}}$ be the itinerary map.





Then S is a continuous semi-conjugacy of $f|_\Lambda$ with the shift. If S is a Cantor set, then S is one-to-one. Is S^{-1} continuous?

Example. $\phi : [0,1) \cup [2,3] \to [0,2]$, $\phi(x) = x$ for $0 \le x < 1$, $\phi(x) = x - 1$ for $2 \le x \le 3$.

The map ϕ is continuous and invertible, but the inverse is not continuous.

Compact sets

Definition. A subset E of a topological space X is called **(sequentially) compact** if any sequence of its elements has a subsequence converging to an element of E.

Remark. There are alternative notions of compactness. They are all the same provided that X is metrizable.

Theorem (Bolzano-Weierstrass) A subset of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition 1 The image of a compact set under a continuous map is also compact.

Proposition 2 Suppose (X, d) and (Y, ρ) are metric spaces, and a continuous map $f : X \to Y$ is invertible. If X is compact, then the inverse map f^{-1} is continuous as well.

Theorem The topological space $\Sigma_{\mathcal{A}}$ of infinite words over a finite alphabet $\mathcal{A}$ is compact.

Proof: Suppose $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \dots$ is a sequence of infinite words over the alphabet $\mathcal{A}$. We need to show that it has a subsequence $\mathbf{s}^{(n_1)}, \mathbf{s}^{(n_2)}, \mathbf{s}^{(n_3)}, \dots$ converging to some $\mathbf{s} \in \Sigma_{\mathcal{A}}$. The latter means that every finite beginning of $\mathbf{s}$ is also a beginning of $\mathbf{s}^{(n_k)}$ for k large enough.

Since $\mathcal{A}$ is a finite set, the number of finite words over $\mathcal{A}$ of any prescribed length is finite. It follows by induction that there exists a sequence of letters $s_1, s_2, \dots$ such that for any $k \in \mathbb{N}$ the finite word $s_1 s_2 \dots s_k$ occurs as a beginning of $\mathbf{s}^{(n)}$ for infinitely many n's. Then we choose indices $n_1 < n_2 < \dots$ so that $s_1 s_2 \dots s_k$ is a beginning of $\mathbf{s}^{(n_k)}$ for $k = 1, 2, \dots$ It follows that $\mathbf{s}^{(n_k)} \to \mathbf{s} = (s_1 s_2 s_3 \dots)$ as $k \to \infty$.

Topological transitivity

Suppose $f : X \to X$ is a continuous transformation of a topological space X.

Definition. The map f is **topologically transitive** if for any nonempty open sets $U, V \subset X$ there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$.

$$U \ni x \longmapsto f(x) \longmapsto f^2(x) \longmapsto \cdots \longmapsto f^n(x) \in V$$

Topological transitivity means that the dynamical system f is, in a sense, indecomposable.

Proposition 1 Topological transitivity is preserved under topological conjugacy.

Proposition 2 If the map f has a dense orbit, then it is topologically transitive provided X is metrizable and has no isolated points.

Proposition 3 If X is a metrizable compact space, then any topologically transitive transformation of X has a dense orbit.

Separation of orbits

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that f has **sensitive dependence on initial conditions** if there is $\delta > 0$ such that, for any $x \in X$ and a neighborhood U of x, there exist $y \in U$ and $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

We say that the map f is **expansive** if there is $\delta > 0$ such that, for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

Proposition If X is compact, then changing the metric d to another metric that induces the same topology cannot affect sensitive dependence on i.c. and expansiveness of the map f.

Corollary For continuous transformations of compact metric spaces, sensitive dependence on initial conditions and expansiveness are preserved under topological conjugacy.

Definition of chaos

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f is **chaotic** if

- f has sensitive dependence on initial conditions;
- f is topologically transitive;
- periodic points of f are dense in X.

The three conditions provide the dynamical system f with unpredictability, indecomposability, and an element of regularity (recurrence).

Examples of chaotic systems

- The shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is chaotic.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map and Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0, 1]$. If Λ is a Cantor set then the restriction $f|_\Lambda$ of the map f to Λ is chaotic (otherwise it is not).

Recall that Λ is a Cantor set if and only if the itinerary map $S : \Lambda \to \Sigma_{\{0,1\}}$ is one-to-one, in which case S is a topological conjugacy of $f|_\Lambda$ with the shift on $\Sigma_{\{0,1\}}$.

The shift

Suppose $\mathcal{A}$ is a finite alphabet consisting of at least 2 letters.

Theorem The shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is chaotic.

We already know that periodic points of the shift are dense in $\Sigma_{\mathcal{A}}$. Also, the shift admits a dense orbit and hence is topologically transitive. It remains to check sensitive dependence on initial conditions.

Lemma The shift is expansive.

Idea of the proof: If $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}$ are distinct infinite words, then for some $n \geq 0$ the shifted words $\sigma^n(\mathbf{s})$ and $\sigma^n(\mathbf{t})$ differ in the first letter.

Finally, expansiveness implies sensitive dependence on initial conditions unless the phase space has isolated points, which is not the case since $\mathcal{A}$ has at least two letters.