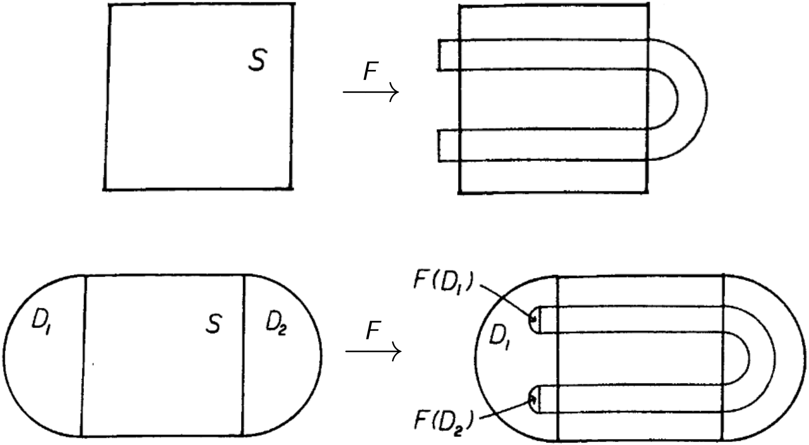
MATH 614

Dynamical Systems and Chaos

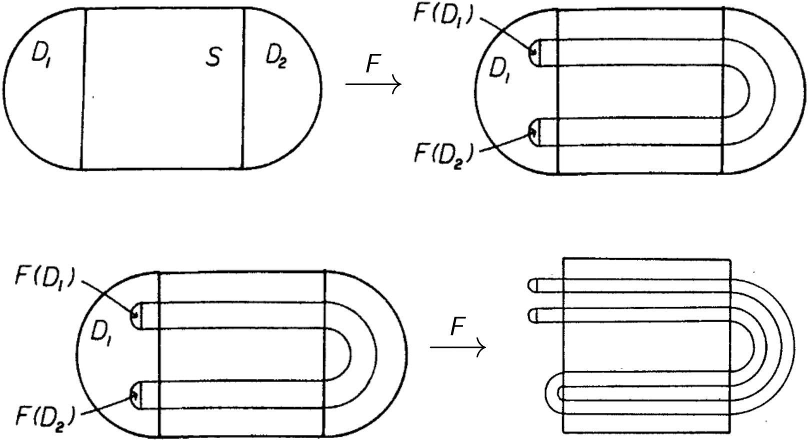
Lecture 19:
The horseshoe map.
Invertible symbolic dynamics.

The Smale horseshoe map

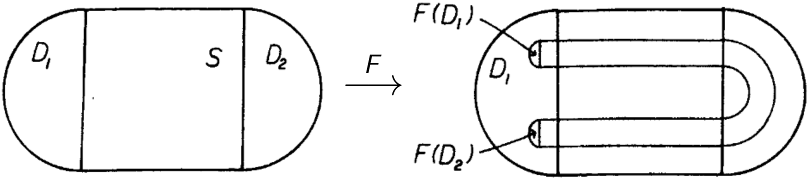
Stephen Smale, 1960



The Smale horseshoe map

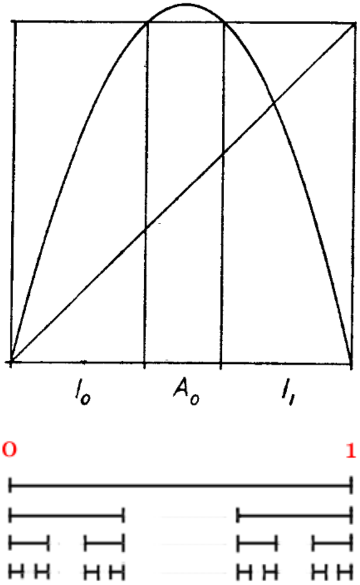


The Smale horseshoe map

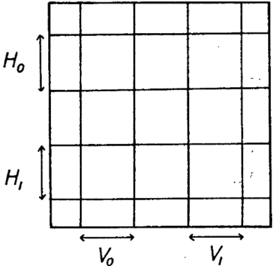


The map F is contracting on D_1 and $F(D_1) \subset D_1$. It follows that there is a unique fixed point $p \in D_1$ and the orbit of any point in D_1 converges to p.

Moreover, any orbit that leaves the square S converges to p.


I_0
A_0
I_1
0
1

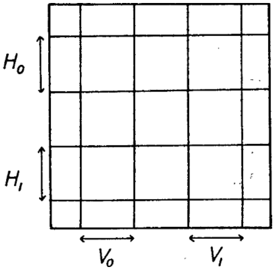
Itineraries



$$F^{-1}(S) = V_0 \cup V_1, \;\; F(V_0) = H_0, \;\; F(V_1) = H_1.$$

Let Λ_1 be the set of all points in S whose orbits stay in S. We have $S = I_H \times I_V$ and $\Lambda_1 = \Xi_1 \times I_V$, where Ξ_1 is a Cantor set. Since $\Lambda_1 \subset V_0 \cup V_1$, we can define the itinerary map $S_+ : \Lambda_1 \to \Sigma_{\{0,1\}}$. This map is continuous and onto. For any infinite word $\mathbf{s} = (s_0 s_1 s_2 \dots)$, the preimage $S_+^{-1}(\mathbf{s})$ is a vertical segment $\{x\} \times I_V$.

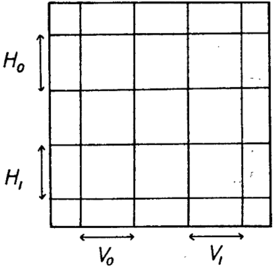
Itineraries



$$F^{-1}(S) = V_0 \cup V_1, \;\; F(V_0) = H_0, \;\; F(V_1) = H_1.$$

Let Λ_2 be the set of all points in S with infinite backward orbit. We have $S = I_H \times I_V$ and $\Lambda_2 = I_H \times \Xi_2$, where Ξ_2 is a Cantor set. Since $\Lambda_2 \subset H_0 \cup H_1$, we can define another itinerary map $S_- : \Lambda_2 \to \Sigma_{\{0,1\}}$ for the inverse map F^{-1}. This itinerary map is also continuous and onto. For any infinite word $\mathbf{t} = (t_0 t_1 t_2 \dots)$, the preimage $S_-^{-1}(\mathbf{t})$ is a horizontal segment $I_H \times \{y\}$.

Itineraries



$$F^{-1}(S) = V_0 \cup V_1, \quad F(V_0) = H_0, \quad F(V_1) = H_1.$$

Finally, let $\Lambda = \Lambda_1 \cap \Lambda_2$. We have $\Lambda = \Xi_1 \times \Xi_2$.

For any $\mathbf{p} \in \Lambda$ we can define the full itinerary $S_{\pm}(\mathbf{p}) = (\dots t_2 t_1 t_0 . s_0 s_1 s_2 \dots)$, where $S_+(\mathbf{p}) = (s_0 s_1 s_2 \dots)$ and $S_-(\mathbf{p}) = (t_0 t_1 t_2 \dots)$. Then

$$S_{\pm}(F(\mathbf{p})) = (\dots t_2 t_1 t_0 s_0 . s_1 s_2 \dots).$$

Indeed, $S_{\pm}(\mathbf{p})$ is the itinerary of the full orbit of **p** under the map F relative to the sets V_0 and V_1.

Bi-infinite words

Given a finite set $\mathcal{A}$ with at least 2 elements (an alphabet), we denote by $\Sigma_{\mathcal{A}}^{\pm}$ the set of all **bi-infinite words** over $\mathcal{A}$, i.e., bi-infinite sequences $\mathbf{s} = (\dots s_{-2}s_{-1}.s_0s_1s_2\dots)$, $s_i \in \mathcal{A}$. Any bi-infinite word in $\Sigma_{\mathcal{A}}^{\pm}$ comes with the standard numbering of letters determined by the decimal point.

For any finite words w_-, w_+ over the alphabet $\mathcal{A}$, we define a **cylinder** $C(w_-, w_+)$ to be the set of all bi-infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}^{\pm}$ of the form $(\dots s_{-2}s_{-1}w_-.w_+s_1s_2\dots)$, $s_i \in \mathcal{A}$. The topology on $\Sigma_{\mathcal{A}}^{\pm}$ is defined so that open sets are unions of cylinders. Two bi-infinite words are considered close in this topology if they have a long common part around the decimal point.

The topological space $\Sigma_{\mathcal{A}}^{\pm}$ is metrizable. A compatible metric is defined as follows. For any $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}^{\pm}$ we let $d(\mathbf{s}, \mathbf{t}) = 2^{-n}$ if $s_i = t_i$ for $0 \le |i| < n$ while $s_n \ne t_n$ or $s_{-n} \ne t_{-n}$. Also, let $d(\mathbf{s}, \mathbf{t}) = 0$ if $\mathbf{s} = \mathbf{t}$.

Invertible symbolic dynamics

The **shift** transformation $\sigma : \Sigma_{\mathcal{A}}^{\pm} \to \Sigma_{\mathcal{A}}^{\pm}$ is defined by $\sigma(\dots s_{-2}s_{-1}.s_0s_1s_2\dots) = (\dots s_{-2}s_{-1}s_0.s_1s_2\dots)$. It is also called the **two-sided shift** while the shift on $\Sigma_{\mathcal{A}}$ is called the **one-sided shift**.

Proposition 1 The two-sided shift is a homeomorphism.

Proposition 2 Periodic points of σ are dense in $\Sigma_{\mathcal{A}}^{\pm}$.

Proposition 3 The two-sided shift admits a dense orbit.

Proposition 4 The two-sided shift is not expansive.

Proposition 5 The two-sided shift is chaotic.

Proposition 6 The itinerary map $S_{\pm} : \Lambda \to \Sigma_{\mathcal{A}}^{\pm}$ of the horseshoe map is a homeomorphism.

Proposition 7 The topological space $\Sigma_{\mathcal{A}}^{\pm}$ is homeomorphic to $\Sigma_{\mathcal{A}}$ and to $\Sigma_{\mathcal{A}} \times \Sigma_{\mathcal{A}}$.