MATH 614

Dynamical Systems and Chaos

Lecture 25:
Chain recurrence.

Chain recurrence

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. A point $x \in X$ is **recurrent** for the map F if for any $\varepsilon > 0$ there is an integer $n > 0$ such that $d\big(F^n(x), x\big) < \varepsilon$. The point x is **chain recurrent** for F if, for any $\varepsilon > 0$, there are points $x_0 = x, x_1, x_2, \dots, x_k = x$ and positive integers $n_1, n_2, \dots, n_k$ such that $d\big(F^{n_i}(x_{i-1}), x_i\big) < \varepsilon$ for $1 \le i \le k$.

A sequence $x_0, x_1, \dots, x_k$ is called an ε-**pseudo-orbit** of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $1 \le i \le k$. The point $x \in X$ is chain recurrent for F if, for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $x_0, x_1, \dots, x_k$ with $x_0 = x_k = x$.

Chain recurrence: properties

- Any periodic point is recurrent.
- Any eventually periodic (but not periodic) point is not recurrent.
- If a point $x \in X$ is chain recurrent under a map $f : X \to X$, then so is $F(x)$.
- Any limit point of any orbit $x, F(x), F^2(x), \ldots$ is chain recurrent. In the case F is invertible, any limit point of any backward orbit $x, F^{-1}(x), F^{-2}(x), \ldots$ is chain recurrent.
- If the orbit of x is dense in X, then x is recurrent unless x is an isolated point in X and not periodic for F.
- The set of all chain recurrent points is closed.
- For a topologically transitive map, all points are chain recurrent.
- Topological conjugacy preserves recurrence and chain recurrence.

- If $x \in W^s(p)$ for a periodic point p, then x is not recurrent unless $x = p$.

- If $X = S^1$ and F is a rotation then every point is recurrent (since either all points are periodic or all orbits are dense).

- If X is the torus $\mathbb{T}^n$ and F is a translation then every point is recurrent (since F preserves distances and volume).

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then every point $\mathbf{s} \in X$ is chain recurrent. Indeed, let $\mathbf{s}^{(n)} = w_n w_n w_n \dots,$ where w_n is the beginning of $\mathbf{s}$ of length n. Then $\sigma^n\big(\mathbf{s}^{(n)}\big) = \mathbf{s}^{(n)}$ and $\mathbf{s}^{(n)} \to \mathbf{s}$ as $n \to \infty$.

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then not every point is recurrent. For example, $\mathbf{s} = (1000\dots)$ is not recurrent.

- If $X = \Sigma_{\mathcal{A}}^{\pm}$ and $F = \sigma$ is the two-sided shift, then every point is chain recurrent but not every point is recurrent, e.g., $\mathbf{s} = (\dots 000.1000\dots)$.

Let $F : X \to X$ be a homeomorphism of a metric space X.

Definition. Suppose $x \in W^s(p) \cap W^u(q)$, where p and q are periodic points of F. Then x is called **heteroclinic** if $p \neq q$ and **homoclinic** if $p = q$.

- Any homoclinic point is chain recurrent.

- If $X = \mathbb{T}^2$ and F is a hyperbolic toral automorphism, then all points of X are chain recurrent (periodic points of F are dense and so are homoclinic points for the fixed point $[0, 0]$).

- If F is the logistic map $F(x) = \mu x(1 - x)$, $\mu > 4$, then chain recurrent points are all points of the invariant Cantor set.

- If F is the solenoid map, then chain recurrent points are all points of the solenoid.

- If F is the horseshoe map, then chain recurrent points are the attracting fixed point and all points of the invariant Cantor set.