MATH 614

Dynamical Systems and Chaos

Lecture 30: Neutral fixed points (continued). The Julia set.

Classification of periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function. Suppose that $F(z_0) = z_0$ for some $z_0 \in U$. The fixed point z_0 is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

Now suppose that $F^n(z_1) = z_1$ for some $z_1 \in U$ and an integer $n \geq 1$. The periodic point z_1 is called

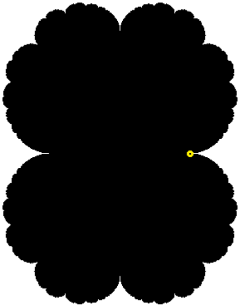
- attracting if $|(F^n)'(z_1)| < 1$;
- repelling if $|(F^n)'(z_1)| > 1$;
- neutral if $|(F^n)'(z_1)| = 1$.

The multiplier $(F^n)'(z_1)$ is the same for all points in the orbit of z_1 (in particular, all these points are of the same type as z_1). Moreover, the multiplier is preserved under any holomorphic change of coordinates.

Neutral fixed points

Example. • $F(z) = z + z^2$.

The map has a fixed point at 0, which is neutral: $F'(0) = 1$. The set D_0 of all points z satisfying $F^n(z) \to 0$ as $n \to \infty$ is open and connected.

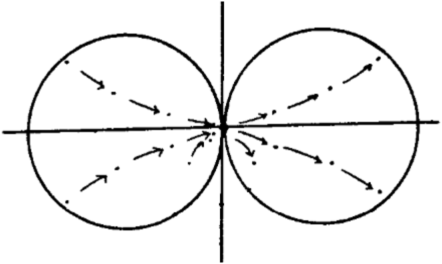


The fixed point 0 is one of the cusp points at the boundary of D_0. The others correspond to eventually fixed points.

Neutral fixed points

Proposition Suppose a function F is holomorphic at 0 and satisfies $F(0) = 0$, $F'(0) = 1$, $F''(0) = 2$ so that $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$.

Then there exists $\mu > 0$ such that **(i)** all points in the disc $D_- = \{z \in \mathbb{C} : |z + \mu| < \mu\}$ are attracted to 0; and **(ii)** all points in the disc $D_+ = \{z \in \mathbb{C} : |z - \mu| < \mu\}$ are repelled from 0.



Proof: We change coordinates using the function $H(z) = 1/z$, which maps the discs D_- and D_+ onto halfplanes $\operatorname{Re} z < -1/(2\mu)$ and $\operatorname{Re} z > 1/(2\mu)$.

The function F is changed to $G(z) = 1/F(1/z)$. Since $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$, it follows that

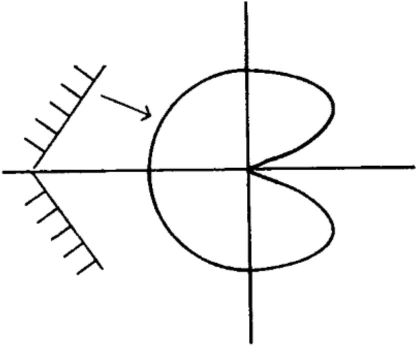
$$\begin{aligned} F(1/z) &= z^{-1} + z^{-2} + O(|z|^{-3}) \\ &= z^{-1}\big(1 + z^{-1} + O(|z|^{-2})\big) \quad \text{as} \quad z \to \infty. \end{aligned}$$

Then

$$\begin{aligned} G(z) &= z\big(1 + z^{-1} + O(|z|^{-2})\big)^{-1} \\ &= z\big(1 - z^{-1} + O(|z|^{-2})\big) = z - 1 + O(|z|^{-1}). \end{aligned}$$

If μ is small enough, then the halfplane $\operatorname{Re} z < -1/(2\mu)$ is invariant under the map G while the halfplane $\operatorname{Re} z > 1/(2\mu)$ is invariant under G^{-1}.

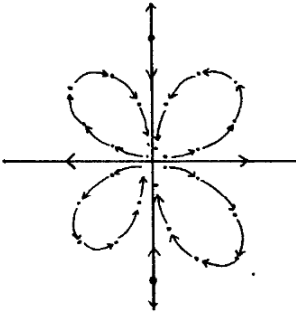
In the proof of the proposition, we could use wedge-shaped regions instead of halfplanes. This would allow to extend basins of attraction from discs to cardioid-shaped regions.

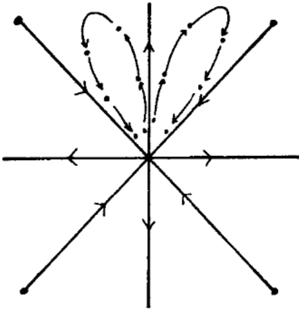


In the case not all points near 0 are attracted to 0, the set of points that are attracted is locally a simply connected domain with 0 on its boundary. This domain is called the **attracting petal** of the fixed point 0. Similarly, there is also the **repelling petal** of 0.

More types of neutral fixed points

$F_1(z) = z + z^3$ $\qquad$ $F_2(z) = z + z^5$





In the first example, there are two attracting and two repelling petals of the fixed point 0. In the second example, there are 4 attracting and 4 repelling petals.

Siegel discs

Theorem (Siegel) Let F be a holomorphic function at z_0 such that $F(z_0) = z_0$ and $F'(z_0) = e^{2\pi i\alpha}$, where α is irrational. Suppose that α is not very well approximated by rational numbers, namely, $|\alpha - p/q| > aq^{-b}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}$. Then there is a neighborhood U of z_0 on which the function F is analytically conjugate to the irrational rotation $L(z) = e^{2\pi i\alpha}z$.

The domain U is called a **Siegel disc**.

In the case α is well approximated by rational numbers, it can happen that the fixed point z_0 is a limit point of other periodic points of the map F.

Idea of the proof: We are looking for a map h of the form $h(z) = z + \sum_{i=2}^{\infty} c_i z^i$, where c_i are unknown coefficients. Let $F(z) = \lambda z + \sum_{i=2}^{\infty} a_i z^i$ be the Taylor expansion of F, $\lambda = e^{2\pi i \alpha}$.

The condition $F \circ h = h \circ L$ holds if

$$\lambda h(z) + \sum_{i=2}^{\infty} a_i (h(z))^i = \lambda z + \sum_{i=2}^{\infty} c_i \lambda^i z^i$$

or, equivalently,

$$\sum_{i=2}^{\infty} (\lambda^i - \lambda) c_i z^i = \sum_{i=2}^{\infty} a_i (h(z))^i.$$

From this equality of formal power series we can recursively determine all coefficients c_i provided that λ is not a root of unity. For example, $c_2 = a_2/(\lambda^2 - \lambda)$. In general, $c_i = K_i(\lambda, a_2, \ldots, a_i, c_2, \ldots, c_{i-1})/(\lambda^i - \lambda)$ for some polynomial K_i.

The Siegel disc exists if and only if the radius of convergence for the power series $h(z)$ is positive.

The Julia set

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Informally, the Julia set of P is the set of points where iterates of P exhibit sensitive dependence on initial conditions (chaotic behavior). The Fatou set of P is the set of points where iterates of P exhibit regular, stable behavior.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P.

Example. $Q_0 : \mathbb{C} \to \mathbb{C}$, $Q_0(z) = z^2$.

0 is an attracting fixed point. The other periodic points are located on the unit circle $|z| = 1$. All of them are repelling.

Any point of the form

$$\exp\Bigl(2\pi i \frac{m}{n}\Bigr),$$

where m, n are integers and n is odd, is periodic. Hence periodic points are dense in the unit circle.

Thus $J(Q_0) = \{z \in \mathbb{C} : |z| = 1\}$.

Quadratic family

The quadratic family $Q_c : \mathbb{C} \to \mathbb{C}$, $c \in \mathbb{C}$,
$Q_c(z) = z^2 + c$.

Theorem $J(Q_{-2}) = [-2, 2]$.

Proof: The map $H(z) = z + z^{-1}$ is holomorphic on $R = \{z \in \mathbb{C} : |z| > 1\}$. It maps R onto $\mathbb{C} \setminus [-2, 2]$ in a one-to-one way (a conformal map). Also, H maps each of the semicircles

$$\{e^{i\phi} \mid 0 \leq \phi \leq \pi\} \quad \text{and} \quad \{e^{i\phi} \mid -\pi \leq \phi \leq 0\}$$

homeomorphically onto $[-2, 2]$.
Finally, $H(Q_0(z)) = Q_{-2}(H(z))$ for all $z \neq 0$.