Math 412-501
Theory of Partial Differential Equations

Lecture 1: Introduction. Heat equation
A **differential equation** is an equation involving an unknown function and certain of its derivatives.

An **ordinary differential equation (ODE)** is an equation involving an unknown function of one variable and certain of its derivatives.

A **partial differential equation (PDE)** is an equation involving an unknown function of two or more variables and certain of its partial derivatives.
Examples

\[ x^2 + 2x + 1 = 0 \]  
(algebraic equation)

\[ f(2x) = 2(f(x))^2 - 1 \]  
(functional equation)

\[ f'(t) + t^2 f(t) = 4 \]  
(ODE)

\[ \frac{\partial u}{\partial x} + 3 \frac{\partial^2 u}{\partial x \partial y} - u \frac{\partial u}{\partial y} \]  
(not an equation)

\[ \frac{\partial u}{\partial x} - 5 \frac{\partial u}{\partial y} = u \]  
(PDE)

\[ u + u^2 = \frac{\partial^2 u}{\partial x \partial y} (0, 0) \]  
(functional-differential equation)
heat equation: \[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}
\]

wave equation: \[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

Laplace’s equation: \[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

In the first two equations, \(u = u(x, t)\). In the latter one, \(u = u(x, y)\).
heat equation: \[
\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

wave equation: \[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Laplace’s equation: \[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0
\]

In the first two equations, \( u = u(x, y, t) \). In the latter one, \( u = u(x, y, z) \).
Heat conduction in a rod

\[ u(x, t) = \text{temperature} \]
\[ e(x, t) = \text{thermal energy density (thermal energy per unit volume)} \]
\[ Q(x, t) = \text{density of heat sources (heat energy per unit volume generated per unit time)} \]
\( \phi(x, t) = \text{heat flux (thermal energy flowing per unit surface per unit time)} \)

\( \phi(x, t) \geq 0 \) if heat energy is flowing to the right,

\( \phi(x, t) \leq 0 \) if heat energy is flowing to the left.
**Conservation of heat energy** (in a volume in a period of time):

<table>
<thead>
<tr>
<th>change of heat energy</th>
<th>=</th>
<th>heat energy flowing across boundary + heat energy generated inside</th>
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</thead>
</table>

<table>
<thead>
<tr>
<th>rate of change of heat energy</th>
<th>=</th>
<th>heat energy flowing across boundary + heat energy generated inside per unit time</th>
</tr>
</thead>
</table>
\( A = \text{area of a section} \)

heat energy = \( e(x, t) \cdot A \cdot \Delta x \)

rate of change of heat energy = \( \frac{\partial}{\partial t} \left( e(x, t) \cdot A \cdot \Delta x \right) \)

heat energy flowing across boundary per unit time 
= \( \phi(x, t) \cdot A - \phi(x + \Delta x, t) \cdot A \)

heat energy generated inside per unit time 
= \( Q(x, t) \cdot A \cdot \Delta x \)
\[
\frac{\partial}{\partial t} \left( e(x, t) \cdot A \cdot \Delta x \right) = \phi(x, t) \cdot A - \phi(x + \Delta x, t) \cdot A \\
+ Q(x, t) \cdot A \cdot \Delta x
\]

\[
\frac{\partial e(x, t)}{\partial t} = \frac{\phi(x, t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x, t)
\]

\[
\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q
\]
\[ c(x) = \text{specific heat or heat capacity (the heat energy supplied to a unit mass of a substance to raise its temperature one unit)} \]

\[ \rho(x) = \text{mass density (mass per unit volume)} \]

*Thermal energy in a volume is equal to the energy it takes to raise the temperature of the volume from a reference temperature (zero) to its actual temperature.*

\[ e(x, t) \cdot A \cdot \Delta x = c(x)u(x, t) \cdot \rho(x) \cdot A \cdot \Delta x \]
\[ e(x, t) = c(x)\rho(x)u(x, t) \]

\[
c\rho \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q
\]

**Fourier’s law of heat conduction:**

\[ \phi = -K_0 \frac{\partial u}{\partial x}, \]

where \( K_0 = K_0(x, u) \) is called the *thermal conductivity*. 
Heat equation:

\[ c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + Q \]

Assuming \( K_0 = \text{const} \), we have

\[ c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q \]

Assuming \( K_0 = \text{const} \), \( c = \text{const} \), \( \rho = \text{const} \) (uniform rod), and \( Q = 0 \) (no heat sources), we obtain

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \]

where \( k = K_0(c\rho)^{-1} \) is called the thermal diffusivity.