Math 412-501
Theory of Partial Differential Equations

Lecture 3-2:
Spectral properties of the Laplacian.
Bessel functions.
Eigenvalue problem:

\[ \nabla^2 \phi + \lambda \phi = 0 \quad \text{in} \quad D, \]

\[ \left( \alpha \phi + \beta \frac{\partial \phi}{\partial n} \right) \bigg|_{\partial D} = 0, \]

where \( \alpha, \beta \) are piecewise continuous real functions on \( \partial D \) such that \( |\alpha| + |\beta| \neq 0 \) everywhere on \( \partial D \).

We assume that the boundary \( \partial D \) is piecewise smooth.
6 spectral properties of the Laplacian

**Property 1.** All eigenvalues are real.

**Property 2.** All eigenvalues can be arranged in the ascending order

\[ \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots \]

so that \( \lambda_n \rightarrow \infty \) as \( n \rightarrow \infty \).

This means that:
- there are infinitely many eigenvalues;
- there is a smallest eigenvalue;
- on any finite interval, there are only finitely many eigenvalues.
Property 3. An eigenvalue $\lambda_n$ may be multiple but its multiplicity is finite.

Moreover, the smallest eigenvalue $\lambda_1$ is simple, and the corresponding eigenfunction $\phi_1$ has no zeros inside the domain $D$.

Property 4. Eigenfunctions corresponding to different eigenvalues are orthogonal relative to the inner product

$$\langle f, g \rangle = \iint_D f(x, y)g(x, y) \, dx \, dy.$$
Property 5. Any eigenfunction $\phi$ can be related to its eigenvalue $\lambda$ through the Rayleigh quotient:

$$\lambda = \frac{-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n} ds + \iint_{D} |\nabla \phi|^2 dx \; dy}{\iint_{D} |\phi|^2 dx \; dy}.$$
Property 6. There exists a sequence $\phi_1, \phi_2, \ldots$ of pairwise orthogonal eigenfunctions that is complete in the Hilbert space $L_2(D)$.

Any square-integrable function $f \in L_2(D)$ is expanded into a series

$$f(x, y) = \sum_{n=1}^{\infty} c_n \phi_n(x, y),$$

that converges in the mean. The series is unique:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

If $f$ is piecewise smooth then the series converges pointwise to $f$ at points of continuity.
Rayleigh quotient

Suppose that $\nabla^2 \phi = -\lambda \phi$ in the domain $D$.

Multiply both sides by $\phi$ and integrate over $D$:

$$\iint_D \phi \nabla^2 \phi \, dx \, dy = -\lambda \iint_D |\phi|^2 \, dx \, dy.$$  

Green’s formula:

$$\iint_D \psi \nabla^2 \phi \, dA = \oint_{\partial D} \psi \frac{\partial \phi}{\partial n} \, ds - \iint_D \nabla \psi \cdot \nabla \phi \, dA$$

This is an analog of integration by parts. Now

$$\oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds - \iint_D |\nabla \phi|^2 \, dx \, dy = -\lambda \iint_D |\phi|^2 \, dx \, dy.$$
It follows that

$$\lambda = \frac{- \int_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds + \iint_{D} |\nabla \phi|^2 \, dx \, dy}{\iint_{D} |\phi|^2 \, dx \, dy}.$$

If $\phi$ satisfies the boundary condition $\phi|_{\partial D} = 0$ or $\frac{\partial \phi}{\partial n}|_{\partial D} = 0$ (or mixed), then the one-dimensional integral vanishes. In particular, $\lambda \geq 0$.

If $\frac{\partial \phi}{\partial n} + \alpha \phi = 0$ on $\partial D$, then

$$- \int_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds = \int_{\partial D} \alpha |\phi|^2 \, ds.$$

In particular, if $\alpha \geq 0$ everywhere on $\partial D$, then $\lambda \geq 0$. 
Self-adjointness

\[ \int \int_D \psi \nabla^2 \phi \, dx \, dy = \int_{\partial D} \psi \frac{\partial \phi}{\partial n} \, ds - \int \int_D \nabla \psi \cdot \nabla \phi \, dx \, dy \]

(Green’s first identity)

\[ \int \int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dx \, dy = \int_{\partial D} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \, ds \]

(Green’s second identity)

If \( \phi \) and \( \psi \) satisfy the same boundary condition

\[ \left( \alpha \phi + \beta \frac{\partial \phi}{\partial n} \right) \bigg|_{\partial D} = \left( \alpha \psi + \beta \frac{\partial \psi}{\partial n} \right) \bigg|_{\partial D} = 0 \]

then \( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} = 0 \) everywhere on \( \partial D \).
If $\phi$ and $\psi$ satisfy the same boundary condition then
\[ \int \int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dx \, dy = 0. \]

If $\phi$ and $\psi$ are complex-valued functions then also
\[ \int \int_D (\phi \nabla^2 \psi - \bar{\psi} \nabla^2 \phi) \, dx \, dy = 0 \]

(because $\nabla^2 \bar{\psi} = \nabla^2 \psi$ and $\bar{\psi}$ satisfies the same boundary condition as $\psi$).

Thus $\langle \nabla^2 \phi, \psi \rangle = \langle \phi, \nabla^2 \psi \rangle$, where
\[ \langle f, g \rangle = \int \int_D f(x, y)\bar{g}(x, y) \, dx \, dy. \]
Eigenvalue problem:

\[ \nabla^2 \phi + \lambda \phi = 0 \quad \text{in} \quad D, \]

\[ \left( \alpha \phi + \beta \frac{\partial \phi}{\partial n} \right) \bigg|_{\partial D} = 0. \]

The Laplacian \( \nabla^2 \) is self-adjoint in the subspace of functions satisfying the boundary condition.

Suppose \( \phi \) is an eigenfunction belonging to an eigenvalue \( \lambda \). Let us show that \( \lambda \in \mathbb{R} \).

Since \( \nabla^2 \phi = -\lambda \phi \), we have that

\[ \langle \nabla^2 \phi, \phi \rangle = \langle -\lambda \phi, \phi \rangle = -\lambda \langle \phi, \phi \rangle, \]

\[ \langle \phi, \nabla^2 \phi \rangle = \langle \phi, -\lambda \phi \rangle = -\bar{\lambda} \langle \phi, \phi \rangle. \]

Now \( \langle \nabla^2 \phi, \phi \rangle = \langle \phi, \nabla^2 \phi \rangle \) and \( \langle \phi, \phi \rangle > 0 \) imply \( \lambda \in \mathbb{R} \).
Suppose $\phi_1$ and $\phi_2$ are eigenfunctions belonging to different eigenvalues $\lambda_1$ and $\lambda_2$.

Let us show that $\langle \phi_1, \phi_2 \rangle = 0$.

Since $\nabla^2 \phi_1 = -\lambda_1 \phi_1$, $\nabla^2 \phi_2 = -\lambda_2 \phi_2$, we have that

\[
\langle \nabla^2 \phi_1, \phi_2 \rangle = \langle -\lambda_1 \phi_1, \phi_2 \rangle = -\lambda_1 \langle \phi_1, \phi_2 \rangle,
\]
\[
\langle \phi_1, \nabla^2 \phi_2 \rangle = \langle \phi_1, -\lambda_2 \phi_2 \rangle = -\lambda_2 \langle \phi_1, \phi_2 \rangle.
\]

But $\langle \nabla^2 \phi_1, \phi_2 \rangle = \langle \phi_1, \nabla^2 \phi_2 \rangle$, hence

\[-\lambda_1 \langle \phi_1, \phi_2 \rangle = -\lambda_2 \langle \phi_1, \phi_2 \rangle.
\]

We already know that $\bar{\lambda}_2 = \lambda_2$. Also, $\lambda_1 \neq \lambda_2$.

It follows that $\langle \phi_1, \phi_2 \rangle = 0$. 

The main purpose of the Rayleigh quotient

Consider a functional (function on functions)

\[ RQ[\phi] = - \oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds + \iint_{D} |\nabla \phi|^2 \, dx \, dy \]

\[ \frac{\iint_{D} |\phi|^2 \, dx \, dy}{\iint_{D} |\phi|^2 \, dx \, dy}. \]

If \( \phi \) is an eigenfunction of \( -\nabla^2 \) in the domain \( D \) with some boundary condition, then \( RQ[\phi] \) is the corresponding eigenvalue.

What if \( \phi \) is not?
Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$ be eigenvalues of a particular eigenvalue problem counted with multiplicities.

That is, a simple eigenvalue appears once in this sequence, an eigenvalue of multiplicity two appears twice, and so on.

There is a complete orthogonal system $\phi_1, \phi_2, \ldots$ in the Hilbert space $L_2(D)$ such that $\phi_n$ is an eigenfunction belonging to $\lambda_n$. 
Theorem (i) $\lambda_1 = \min RQ[\phi]$, where the minimum is taken over all nonzero functions $\phi$ which are differentiable in $D$ and satisfy the boundary condition. Moreover, if $RQ[\phi] = \lambda_1$ then $\phi$ is an eigenfunction.

(ii) $\lambda_n = \min RQ[\phi]$, where the minimum is taken over all nonzero functions $\phi$ which are differentiable in $D$, satisfy the boundary condition, and such that $\langle \phi, \phi_k \rangle = 0$ for $1 \leq k < n$. Moreover, the minimum is attained only on eigenfunctions.

Main idea of the proof: $RQ[\phi] = \frac{\langle -\nabla^2 \phi, \phi \rangle}{\langle \phi, \phi \rangle}$.

(see Haberman 5.6)
Spectral properties of the Laplacian in a circle

Eigenvalue problem:
\[ \nabla^2 \phi + \lambda \phi = 0 \quad \text{in} \quad D = \{(x, y) : x^2 + y^2 \leq R^2\}, \]
\[ u|_{\partial D} = 0. \]

In polar coordinates \((r, \theta)\):
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \lambda \phi = 0 \]
\[ (0 < r < R, -\pi < \theta < \pi), \]
\[ \phi(R, \theta) = 0 \quad (-\pi < \theta < \pi). \]
Additional boundary conditions:

\[ |\phi(0, \theta)| < \infty \quad (-\pi < \theta < \pi), \]

\[ \phi(r, -\pi) = \phi(r, \pi), \quad \frac{\partial \phi}{\partial \theta}(r, -\pi) = \frac{\partial \phi}{\partial \theta}(r, \pi) \quad (0 < r < R). \]

Separation of variables: \( \phi(r, \theta) = f(r)h(\theta). \)

Substitute this into the equation:

\[ f''(r)h(\theta) + r^{-1}f'(r)h(\theta) + r^{-2}f(r)h''(\theta) + \lambda f(r)h(\theta) = 0. \]

Divide by \( f(r)h(\theta) \) and multiply by \( r^2: \)

\[ r^2 f''(r) + r f'(r) + \lambda r^2 f(r) \frac{h''(\theta)}{h(\theta)} = 0. \]
It follows that
\[ r^2 f''(r) + rf'(r) + \lambda r^2 f(r) \frac{f''(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \mu = \text{const.} \]

The variables have been separated:
\[ r^2 f'' + rf' + (\lambda r^2 - \mu) f = 0, \]
\[ h'' = -\mu h. \]

Boundary conditions \( \phi(R, \theta) = 0 \) and \( |\phi(0, \theta)| < \infty \)
hold if \( f(R) = 0 \) and \( |f(0)| < \infty \).

Boundary conditions \( \phi(r, -\pi) = \phi(r, \pi) \) and
\( \frac{\partial \phi}{\partial \theta}(r, -\pi) = \frac{\partial \phi}{\partial \theta}(r, \pi) \)
hold if \( h(-\pi) = h(\pi) \) and
\( h'(-\pi) = h'(\pi) \).
Eigenvalue problem:

\[ h'' = -\mu h, \quad h(-\pi) = h(\pi), \quad h'(-\pi) = h'(\pi). \]

Eigenvalues: \( \mu_m = m^2, \quad m = 0, 1, 2, \ldots \)

\( \mu_0 = 0 \) is simple, the others are of multiplicity 2.

Eigenfunctions: \( h_0 = 1, \quad h_m(\theta) = \cos m\theta \) and \( \tilde{h}_m(\theta) = \sin m\theta \) for \( m \geq 1 \).
Dependence on \( r \):
\[
  r^2 f'' + rf' + (\lambda r^2 - \mu) f = 0, \quad f(R) = 0, \quad |f(0)| < \infty.
\]

We may assume that \( \mu = m^2 \), \( m = 0, 1, 2, \ldots \).

Also, we know that \( \lambda > 0 \) (Rayleigh quotient!).

New variable \( z = \sqrt{\lambda} \cdot r \) removes dependence on \( \lambda \):

\[
  z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0.
\]

This is **Bessel’s differential equation** of order \( m \).

Solutions are called **Bessel functions** of order \( m \).