Notes on the commutator group of the group of interval exchange transformations

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Abstract

We study the group of interval exchange transformations and obtain several characterizations of its commutator group. In particular, it turns out that the commutator group is generated by elements of order 2.

1 Introduction

Let (p, q) be an interval of the real line and $p_0 = p < p_1 < \cdots < p_{k-1} < p_k = q$ be a finite collection of points that split (p, q) into subintervals (p_{i-1}, p_i), $i = 1, 2, \ldots, k$. A transformation f of the interval (p, q) that rearranges the subintervals by translation is called an *interval exchange transformation* (see Figure 1). To be precise, the restriction of f to any (p_{i-1}, p_i) is a translation, the translated subinterval remains within (p, q) and does not overlap with the other translated subintervals. The definition is still incomplete as values of f at the points p_i are not specified. The standard way to do this, which we adopt, is to require that f be right continuous. That is, we consider the half-closed interval $I = [p, q)$ partitioned into smaller half-closed intervals $I_i = [p_{i-1}, p_i)$. The interval exchange transformation f is to translate each I_i so that the images $f(I_1), f(I_2), \ldots, f(I_k)$ form another partition of I.

Let λ be a k-dimensional vector whose coordinates are lengths of the intervals $I_1, I_2, \ldots, I_k$. Let π be a permutation on $\{1, 2, \ldots, k\}$ that tells how the intervals are rearranged by f. Namely, $\pi(i)$ is the position of $f(I_i)$ when the the intervals $f(I_1), \ldots, f(I_k)$ are ordered from left to right. For the example in Figure 1, $\pi = (1\,2\,4\,3)$. We refer to the pair (λ, π) as a combinatorial description of f. Given an integer $k \geq 1$, a k-dimensional vector λ with positive coordinates that add up to the length of I, and a permutation π on $\{1, 2, \ldots, k\}$, the pair (λ, π) determines a unique interval exchange transformation of I. The converse is not true. Any partition of I into subintervals that are translated by f gives rise to a distinct combinatorial description. Clearly, such a partition is not unique. However there is a unique partition with the smallest possible number of subintervals.

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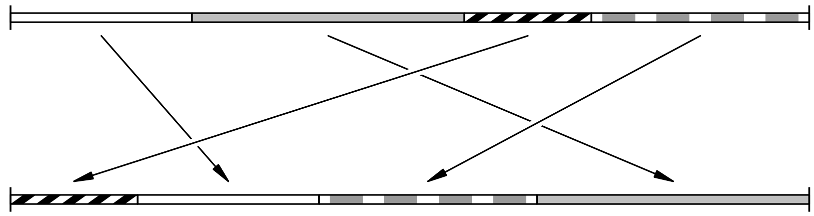

Figure 1: Interval exchange transformation.

The interval exchange transformations have been popular objects of study in ergodic theory. First of all, the exchange of two intervals is equivalent to a rotation of the circle (it becomes one when we identify the endpoints of the interval I thus producing a circle). The exchanges of three or more intervals were first considered by Katok and Stepin [2]. The systematic study started since the paper by Keane [3] who coined the term. For an account of the results, see the survey by Viana [6].

All interval exchange transformations of a fixed interval $I = [p, q)$ form a transformation group $\mathcal{G}_I$. Changing the interval, one obtains an isomorphic group. Indeed, let $J = [p', q')$ be another interval and h be an affine map of I onto J. Then $f \in G_J$ if and only if $h^{-1}fh \in G_I$. We refer to any of the groups G_I as the *group of interval exchange transformations*. The present notes are concerned with group-theoretic properties of G_I. An important tool here is the *scissors congruence invariant* or the *Sah-Arnoux-Fathi (SAF) invariant* of $f \in G_I$ introduced independently by Sah [4] and Arnoux and Fathi [1]. The invariant can be informally defined by

$$\mathrm{Inv}(f) = \int_I 1 \otimes \big(f(x) - x\big)\, dx$$

(the integral is actually a finite sum). The importance stems from the fact that Inv is a homomorphism of the group G_I onto $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$. As a consequence, the invariant vanishes on the commutator group, which is a subgroup of G_I generated by commutators $f^{-1}g^{-1}fg$, where f and g run over G_I.

In this paper, we establish the following properties of the commutator group of the group of interval exchange transformations G_I.

Theorem 1.1 *The following four groups are the same:*

- *the group of interval exchange transformations with zero SAF invariant,*
- *the commutator group of the group of interval exchange transformations,*
- *the group generated by interval exchange transformations of order* 2*,*
- *the group generated by interval exchange transformations of finite order.*

Theorem 1.2 *The quotient of the group of interval exchange transformations by its commutator group is isomorphic to* $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.

Theorem 1.3 *The commutator group of the group of interval exchange transformations is simple.*

It has to be noted that most of these results are already known. A theorem by Sah reproduced in Veech's paper [5] contains Theorems 1.2, 1.3, and part of Theorem 1.1. Unfortunately, the preprint of Sah [4] was never published. Hence we include complete proofs. A new result of the present paper is that the commutator group of $\mathcal{G}_I$ is generated by elements of order 2. This is the central result of the paper as our proofs of the theorems are based on the study of elements of order 2.

The paper is organized as follows. Section 2 contains some elementary constructions that will be used in the proofs of the theorems. The scissors congruence invariant is considered in Section 3. Section 4 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in the same section. Section 5 is devoted to the proof of Theorem 1.3.

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2 Elementary constructions

Let us choose an arbitrary interval $I = [p, q)$. In what follows, all interval exchange transformations are assumed to be defined on I. Also, all subintervals of I are assumed to be half-closed intervals of the form $[x, y)$.

The proofs of Theorems 1.1 and 1.3 are based on several elementary constructions described in this section. First of all, we introduce two basic types of transformations used in those constructions. An *interval swap map of type* a is an interval exchange transformation that interchanges two nonoverlapping intervals of length a by translation while fixing the rest of the interval I. A *restricted rotation of type* (a, b) is an interval exchange transformation that exchanges two neighboring intervals of lengths a and b (the interval of length a must be to the left of the interval of length b) while fixing the rest of I. The type of an interval swap map is determined uniquely, and so is the type of a restricted rotation. Clearly, any interval swap map is an involution. The inverse of a restricted rotation of type (a, b) is a restricted rotation of type (b, a). Any restricted rotation of type (a, a) is also an interval swap map of type a.

Lemma 2.1 *Any interval exchange transformation* f *is a product of restricted rotations. Moreover, if* f *exchanges at least two intervals and* $\mathcal{L}$ *is the set of their lengths, it is enough to use restricted rotations of types* (a, b) *such that* $a, b \in \mathcal{L}$.

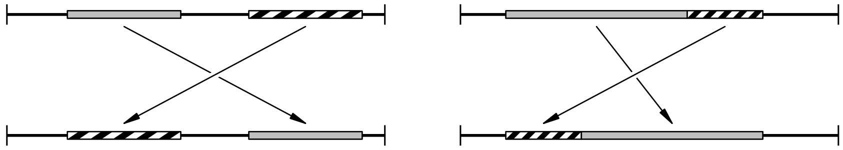

Figure 2: An interval swap map and a restricted rotation.

Proof. The exchange of one interval is the identity. In this case, take any restricted rotation h. Then $f = hh^{-1}$, which is a product of restricted rotations. Now assume that f exchanges $k \geq 2$ intervals. Let $I_1, I_2, \dots, I_k$ be the intervals, ordered from left to right, and (λ, π) be the corresponding combinatorial description of f. Then $\mathcal{L}$ is the set of coordinates of the vector λ.

For any permutation σ on $\{1, 2, \dots, k\}$, let f_σ denote a unique interval exchange transformation with the combinatorial description (λ, σ). Given two permutations σ and τ on $\{1, 2, \dots, k\}$, let $g_{\sigma,\tau} = f_\sigma f_\tau^{-1}$. For any i, $1 \leq i \leq k$, the transformation $g_{\sigma,\tau}$ translates the interval $f_\tau(I_i)$ onto $f_\sigma(I_i)$. It follows that $g_{\sigma,\tau}$ has combinatorial description $(\lambda', \sigma\tau^{-1})$, where $\lambda'_i = \lambda_{\tau^{-1}(i)}$ for $1 \leq i \leq k$. Now suppose that π is expanded into a product of permutations $\pi = \sigma_1\sigma_2 \dots \sigma_m$. For any j, $1 \leq j \leq m$, let $\pi_j = \sigma_j\sigma_{j+1} \dots \sigma_m$. Then $f = h_1h_2 \dots h_m$, where $h_m = f_{\sigma_m}$ and $h_j = g_{\pi_j,\pi_{j+1}}$ for $1 \leq j < m$. By the above each h_j has combinatorial description $(\lambda^{(j)}, \sigma_j)$, where the vector $\lambda^{(j)}$ is obtained from λ by permutation of its coordinates. In the case σ_j is a transposition of neighboring numbers, $\sigma_j = (i,\, i+1)$, the transformation h_j is a restricted rotation of type $\big(\lambda_i^{(j)}, \lambda_{i+1}^{(j)}\big)$. Notice that $\lambda_i^{(j)}, \lambda_{i+1}^{(j)} \in \mathcal{L}$.

It remains to observe that any permutation π on $\{1, 2, \dots, k\}$ is a product of transpositions of neighboring numbers. Indeed, we can represent π as a product of cycles. Further, any cycle $(n_1\, n_2\, \dots\, n_m)$ of length $m \geq 2$ is a product of $m-1$ transpositions: $(n_1\, n_2\, \dots\, n_m) = (n_1\, n_2)(n_2\, n_3) \dots (n_{m-1}\, n_m)$. A cycle of length 1 is the identity, hence it equals $(1\,2)(1\,2)$. Finally, any transposition $(n\, l)$, $n < l$, is expanded into a product of transpositions of neighboring numbers: $(n\, l) = \tau_n\tau_{n+1} \dots \tau_{l-2}\tau_{l-1}\tau_{l-2} \dots \tau_{n+1}\tau_n$, where $\tau_i = (i,\, i+1)$. ∎

Notice that the set $\mathcal{L}$ in Lemma 2.1 depends on the combinatorial description of the interval exchange transformation f. The lemma holds for every version of this set.

Lemma 2.2 *Any interval exchange transformation of finite order is a product of interval swap maps.*

Proof. Suppose $J_1, J_2, \dots, J_k$ are nonoverlapping intervals of the same length a contained in an interval I. Let g be an interval exchange transformation of I that translates J_i onto J_{i+1} for $1 \leq i < k$, translates J_k onto J_1, and fixes the rest of I. If $k \geq 2$ then g is the product of $k-1$ interval swap maps of type a. Namely, $g = h_1h_2 \dots h_{k-1}$, where each h_i interchanges J_i with J_{i+1} by translation while fixing the rest of I. In the case $k = 1$, g is the identity. Then $g = h_0 h_0$ for any interval swap map h_0 on I.

Let f be an interval exchange transformation of I that has finite order. Since there are only finitely many distinct powers of f, there are only finitely many points in I at which one of the powers has a discontinuity. Let $I = I_1 \cup I_2 \cup \ldots \cup I_m$ be a partition of I into subintervals created by all such points. By construction, the restriction of f to any I_i is a translation and, moreover, the translated interval $f(I_i)$ is contained in another element of the partition. Since the same applies to the inverse f^{-1}, it follows that $f(I_i)$ actually coincides with some element of the partition. Hence f permutes the intervals $I_1, I_2, \ldots, I_m$ by translation. Therefore these intervals can be relabeled as J_{ij}, $1 \leq i \leq l$, $1 \leq j \leq k_i$ (l and $k_1, \ldots, k_l$ are some positive integers), so that f translates each J_{ij} onto $J_{i,j+1}$ if $j < k_i$ and onto J_{i1} if $j = k_i$. For any $i \in \{1, 2, \ldots, l\}$ let g_i be an interval exchange transformation that coincides with f on the union of intervals J_{ij}, $1 \leq j \leq k_i$, and fixes the rest of I. It is easy to observe that the transformations $g_1, \ldots, g_l$ commute and $f = g_1 g_2 \ldots g_l$. By the above each g_i can be represented as a product of interval swap maps. Hence f is a product of interval swap maps as well. ■

Lemma 2.3 *Any interval swap map is a commutator of two interval exchange transformations of order* 2.

Proof. Let f be an interval swap map of type a. Let $I_1 = [x, x+a)$ and $I_2 = [y, y+a)$ be nonoverlapping intervals interchanged by f. We split the interval I_1 into two subintervals $I_{11} = [x, x+a/2)$ and $I_{12} = [x+a/2, x+a)$. Similarly, I_2 is divided into $I_{21} = [y, y+a/2)$ and $I_{22} = [y+a/2, y+a)$. Now we introduce three interval swap maps of type $a/2$: g_1 interchanges I_{11} with I_{12}, g_2 interchanges I_{21} with I_{22}, and g_3 interchanges I_{11} with I_{21}. The maps f, g_1, g_2, g_3 permute the intervals $I_{11}, I_{12}, I_{21}, I_{22}$ by translation and fix the rest of the interval I. It is easy to see that $g_1 g_2 = g_2 g_1$. Hence $g = g_1 g_2$ is an element of order 2. Further, we check that $g_3 g = g_3 g_2 g_1$ maps I_{11} onto I_{12}, I_{12} onto I_{21}, I_{21} onto I_{22}, and I_{22} onto I_{11}. Therefore the second iteration $(g_3 g)^2$ interchanges I_{11} with I_{21} and I_{12} with I_{22}, which is exactly how f acts. Thus $f = (g_3 g)^2 = g_3^{-1} g^{-1} g_3 g$. ■

For the next two constructions, we need another definition. The *support* of an interval exchange transformation f is the set of all points in I moved by f. It is the union of a finite number of (half-closed) intervals. For instance, the support of a restricted rotation of type (a, b) is a single interval of length $a+b$. The support of an interval swap map of type a is the union of two nonoverlapping intervals of length a. Note that any interval swap map is uniquely determined by its type and support. The same holds true for any restricted rotation.

Lemma 2.4 *Let f_1 and f_2 be interval swap maps of the same type. If the supports of f_1 and f_2 do not overlap then there exists an interval exchange transformation g of order* 2 *such that $f_2 = g f_1 g$.*

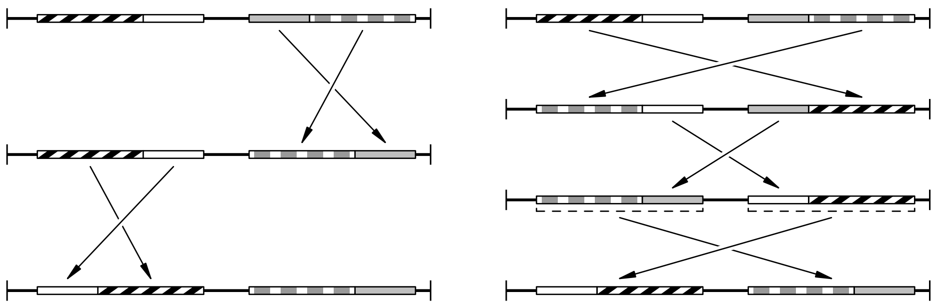

Figure 3: Proof of Lemma 2.5.

Proof. Let a be the type of f_1 and f_2. Let I_1 and I_1' be nonoverlapping intervals of length a interchanged by f_1. Let I_2 and I_2' be nonoverlapping intervals of length a interchanged by f_2. Assume that the supports of f_1 and f_2 do not overlap, i.e., the intervals I_1, I_1', I_2, I_2' do not overlap with each other. Let us introduce two more interval swap maps of type a: g_1 interchanges I_1 with I_2 and g_2 interchanges I_1' with I_2'. Since the supports of g_1 and g_2 do not overlap, the transformations commute. Hence the product $g = g_1 g_2$ is an element of order 2. The maps f_1, f_2, and g permute the intervals I_1, I_1', I_2, I_2' by translation and fix the rest of the interval I. One easily checks that $f_2 = g f_1 g$. ■

Lemma 2.5 *Let f_1 and f_2 be restricted rotations of the same type. If the supports of f_1 and f_2 do not overlap then $f_1^{-1} f_2$ is the product of three interval swap maps.*

Proof. Let (a, b) be the type of f_1 and f_2. Let $I_1 = [x, x+a+b)$ be the support of f_1 and $I_2 = [y, y+a+b)$ be the support of f_2. The transformation f_2 translates the interval $I_{21} = [y, y+a)$ by b and the interval $I_{22} = [y+a, y+a+b)$ by $-a$. The inverse f_1^{-1} is a restricted rotation of type (b, a) with the same support as f_1. It translates the interval $I_{11} = [x, x+b)$ by a and the interval $I_{12} = [x+b, x+a+b)$ by $-b$.

Assume that the supports I_1 and I_2 do not overlap. Let g_1 be the interval swap map of type a that interchanges the intervals I_{12} and I_{21}, let g_2 be the interval swap map of type b that interchanges I_{11} and I_{22}, and let g_3 be the interval swap map of type $a+b$ that interchanges I_1 and I_2. It is easy to check that $f_1^{-1} f_2 = g_3 g_2 g_1 = g_3 g_1 g_2$ (see Figure 3). ■

Lemma 2.6 *Suppose f is a restricted rotation of type (a, b), where $a > b$. Then there exist interval swap maps g_1 and g_2 such that $g_1 f = f g_2$ is a restricted rotation of type $(a-b, b)$.*

Proof. Let $I_0 = [x, x+a+b)$ be the support of f. We define three more transformations with supports inside I_0: g_1 is an interval swap map of type b that interchanges the intervals $[x, x+b)$ and $[x+a, x+a+b)$, g_2 is an interval swap map of type b that interchanges $[x+a-b, x+a)$ and $[x+a, x+a+b)$, and h is a restricted rotation of type $(a-b, b)$ with support $[x, x+a)$. Let us check that $g_1 h = f$. Since $a > b$, the points $x+a-b$ and $x+a$ divide I_0 into three subintervals $I_1 = [x, x+a-b)$, $I_2 = [x+a-b, x+a)$, and $I_3 = [x+a, x+a+b)$. The map h translates I_1 by b, I_2 by $b-a$, and fixes I_3. Then the map g_1 translates I_3 by $-a$, $[x, x+b) = h(I_2)$ by a, and fixes $[x+b, x+a) = h(I_1)$. Therefore the product $g_1 h$ translates I_1 by b, I_2 by b, and I_3 by $-a$. This is exactly how f acts. Similarly, we check that $f = hg_2$. It remains to notice that $g_1 f = g_1^2 h = h = hg_2^2 = fg_2$. ∎

Lemma 2.7 *Let f be a nontrivial interval exchange transformation. Then there exist $\epsilon_0 > 0$ and, for any $0 < \epsilon < \epsilon_0$, interval swap maps g_1, g_2 such that $g_2 f^{-1} g_1 f g_1 g_2$ is an interval swap map of type ϵ.*

Proof. Since f is not the identity, we can find an interval $J = [x, y)$ such that f translates J by some $t \neq 0$. Let $\epsilon_0 = \min(y-x, |t|)$. Given any ϵ, $0 < \epsilon < \epsilon_0$, we introduce two intervals $I_0 = [x, x+\epsilon)$ and $I_1 = [x+t, x+t+\epsilon)$. By construction, I_0 and I_1 do not overlap. Besides, f translates I_0 onto I_1. Let g_0 be an interval swap map of type $\epsilon/2$ that interchanges two halves $I_{01} = [x, x+\epsilon/2)$ and $I_{02} = [x+\epsilon/2, x+\epsilon)$ of the interval I_0. Let g_1 be an interval swap map of type $\epsilon/2$ that interchanges two halves $I_{11} = [x+t, x+t+\epsilon/2)$ and $I_{12} = [x+t+\epsilon/2, x+t+\epsilon)$ of I_1. Since f translates I_0 onto I_1, it follows that $g_0 = f^{-1} g_1 f$. Further, let g_2 be an interval swap map of type $\epsilon/2$ that interchanges I_{02} with I_{11}. The maps g_0, g_1, g_2 permute the nonoverlapping intervals $I_{01}, I_{02}, I_{11}, I_{12}$ by translation and fix the rest of the interval I. It is easy to check that $g_2 g_0 g_1 g_2 = g_2 f^{-1} g_1 f g_1 g_2$ is an interval swap map of type ϵ that interchanges I_0 with I_1. ∎

3 Scissors congruence invariant

Let us recall the construction of the tensor product. Suppose V and W are vector spaces over a field F. Let $Z(V, W)$ be a vector space over F with basis $\{z[v, w]\}_{(v,w) \in V \times W}$. Let $Y(V, W)$ denote the subspace of $Z(V, W)$ spanned by all vectors of the form $z[v_1+v_2, w] - z[v_1, w] - z[v_2, w]$, $z[v, w_1+w_2] - z[v, w_1] - z[v, w_2]$, $z[\alpha v, w] - \alpha z[v, w]$, and $z[v, \alpha w] - \alpha z[v, w]$, where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\alpha \in F$. The *tensor product* of the spaces V and W over the field F, denoted $V \otimes_F W$, is the quotient of the vector space $Z(V, W)$ by $Y(V, W)$. For any $v \in V$ and $w \in W$ the coset $z[v, w] + Y(V, W)$ is denoted $v \otimes w$. By construction, $(v, w) \mapsto v \otimes w$ is a bilinear mapping on $V \times W$. In the case $V = W$, for any vectors $v, w \in V$ we define the *wedge product* $v \wedge w = v \otimes w - w \otimes v$. The subspace of $V \otimes_F V$ spanned by all wedge products is denoted $V \wedge_F V$. By construction, $(v, w) \mapsto v \wedge w$ is a bilinear, skew-symmetric mapping on $V \times V$.

Lemma 3.1 *Suppose V is a vector space over a field F and $v_1, v_2, \dots, v_k \in V$ are linearly independent vectors. Then the wedge products $v_i \wedge v_j$, $1 \le i < j \le k$, are linearly independent in $V \wedge_F V$.*

Proof. For any bilinear function $\omega : V \times V \to F$ let $\tilde{\omega}$ denote a unique linear function on $Z(V,V)$ such that $\tilde{\omega}(z[v,w]) = \omega(v,w)$ for all $v,w \in V$. Since ω is bilinear, the function $\tilde{\omega}$ vanishes on the subspace $Y(V,V)$. Hence it gives rise to a linear function $\hat{\omega} : V \otimes_F V \to F$. By construction, $\hat{\omega}(v \otimes w) = \omega(v,w)$ for all $v,w \in V$.

Let us extend the set $\{v_1, v_2, \dots, v_k\}$ to a basis S for the vector space V. For any $l,m \in \{1,2,\dots,k\}$ we denote by ω_{lm} a unique bilinear function on $V \times V$ such that $\omega_{lm}(v,w) = 1$ if $(v,w) = (v_l, v_m)$ and $\omega_{lm}(v,w) = 0$ for any other pair $(v,w) \in S \times S$. The function ω_{lm} gives rise to a linear function $\hat{\omega}_{lm}$ on $V \otimes_F V$ as described above. For any $i,j \in \{1,2,\dots,k\}$, $i \neq j$, we have $\hat{\omega}_{lm}(v_i \wedge v_j) = 1$ if $i = l$ and $j = m$, $\hat{\omega}_{lm}(v_i \wedge v_j) = -1$ if $i = m$ and $j = l$, and $\hat{\omega}_{lm}(v_i \wedge v_j) = 0$ otherwise.

Consider an arbitrary linear combination

$$\xi = \sum\nolimits_{1 \le i < j \le k} r_{ij} (v_i \wedge v_j)$$

with coefficients r_{ij} from F. It is easy to observe that $\hat{\omega}_{lm}(\xi) = r_{lm}$ for any $1 \le l < m \le k$. Therefore $\xi \neq 0$ unless all r_{ij} are zeros. Thus the wedge products $v_i \wedge v_j$, $1 \le i < j \le k$, are linearly independent over F. ■

Let f be an interval exchange transformation of an interval $I = [p,q)$. Consider an arbitrary partition of I into subintervals, $I = I_1 \cup I_2 \cup \ldots \cup I_k$, such that the restriction of f to any I_i is a translation by some t_i. Let λ_i be the length of I_i, $1 \le i \le k$. The *scissors congruence invariant*, also known as the *Sah-Arnoux-Fathi (SAF) invariant*, of f is

$$\mathrm{Inv}(f) = \lambda_1 \otimes t_1 + \lambda_2 \otimes t_2 + \cdots + \lambda_k \otimes t_k$$

regarded as an element of the tensor product $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$. One can easily check that $\mathrm{Inv}(f) = a \wedge b$ for any restricted rotation f of type (a,b) and $\mathrm{Inv}(g) = 0$ for any interval swap map g. The term 'scissors congruence invariant' is partially explained by the following lemma.

Lemma 3.2 *The scissors congruence invariant* $\mathrm{Inv}(f)$ *of an interval exchange transformation* f *does not depend on the combinatorial description of* f*.*

Proof. Let $I = I_1 \cup \ldots \cup I_k$ be a partition of the interval I into subintervals such that the restriction of f to any I_i is a translation by some t_i. Let $I = I'_1 \cup \ldots \cup I'_m$ be another partition into subintervals such that the restriction of f to any I'_j is a translation by some t'_j. Let λ_i denote the length of I_i ($1 \le i \le k$) and λ'_j denote the length of I'_j ($1 \le j \le m$). We have to show that $\xi = \lambda_1 \otimes t_1 + \cdots + \lambda_k \otimes t_k$ coincides with $\xi' = \lambda'_1 \otimes t'_1 + \cdots + \lambda'_m \otimes t'_m$ in $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$.

For any $1 \le i \le k$ and $1 \le j \le m$ the intersection $I_i \cap I'_j$ is either an interval or the empty set. We let μ_{ij} be the length of the interval in the former case and $\mu_{ij} = 0$ otherwise. Further, let

$$\eta = \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{m} \mu_{ij} \otimes t_i, \qquad \eta' = \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{m} \mu_{ij} \otimes t'_j.$$

Clearly, $t_i = t'_j$ whenever $I_i \cap I'_j$ is an interval. Otherwise $\mu_{ij} = 0$ and $0 \otimes t_i = 0 = 0 \otimes t'_j$. In any case, $\mu_{ij} \otimes t_i = \mu_{ij} \otimes t'_j$. Therefore $\eta = \eta'$. For any $i \in \{1, 2, \dots, k\}$, nonempty intersections $I_i \cap I'_j$, $1 \le j \le m$, form a partition of the interval I_i into subintervals. Hence $\lambda_i = \mu_{i1} + \mu_{i2} + \dots + \mu_{im}$. It follows that $\eta = \xi$. Similarly, we obtain that $\eta' = \xi'$. Thus $\xi = \eta = \eta' = \xi'$. ■

In view of Lemma 3.2, for any interval $I = [p, q)$ we can consider the invariant Inv as a function on $\mathcal{G}_I$, the set of all interval exchange transformations of I.

Lemma 3.3 *The scissors congruence invariant* Inv *is a homomorphism of the group* $\mathcal{G}_I$ *to* $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$.

Proof. Consider arbitrary interval exchange transformations f and g of the interval I. We have to show that $\mathrm{Inv}(fg) = \mathrm{Inv}(f) + \mathrm{Inv}(g)$. Let $I = I_1 \cup I_2 \cup \dots \cup I_k$ be a partition of I into subintervals such that the restrictions of both g and fg to any I_i are translations by some t_i and t'_i, respectively. Let λ_i be the length of I_i, $1 \le i \le k$. Then

$$\begin{aligned}\mathrm{Inv}(g) &= \lambda_1 \otimes t_1 + \lambda_2 \otimes t_2 + \dots + \lambda_k \otimes t_k,\\ \mathrm{Inv}(fg) &= \lambda_1 \otimes t'_1 + \lambda_2 \otimes t'_2 + \dots + \lambda_k \otimes t'_k.\end{aligned}$$

It is easy to see that for any $1 \le i \le k$ the image $g(I_i)$ is an interval of length λ_i and the restriction of f to $g(I_i)$ is the translation by $t'_i - t_i$. Besides, the intervals $g(I_1), g(I_2), \dots, g(I_k)$ form another partition of I. It follows that

$$\mathrm{Inv}(f) = \lambda_1 \otimes (t'_1 - t_1) + \lambda_2 \otimes (t'_2 - t_2) + \dots + \lambda_k \otimes (t'_k - t_k).$$

Since $\lambda_i \otimes (t'_i - t_i) + \lambda_i \otimes t_i = \lambda_i \otimes t'_i$ for all $1 \le i \le k$, we obtain that $\mathrm{Inv}(fg) = \mathrm{Inv}(f) + \mathrm{Inv}(g)$. ■

In the remainder of this section we show that Inv is actually a homomorphism of $\mathcal{G}_I$ onto $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.

Lemma 3.4 *For any $a, b, \epsilon > 0$ there exist pairs of positive numbers (a_1, b_1), $(a_2, b_2), \dots, (a_n, b_n)$ such that*

- $(a_1, b_1) = (a, b)$,
- $(a_{i+1}, b_{i+1}) = (a_i - b_i, b_i)$ *or* $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ *for* $1 \le i \le n - 1$,
- $a_n + b_n < \epsilon$ *or* $a_n = b_n$.

Proof. We define a finite or infinite sequence of pairs inductively. First of all, $(a_1, b_1) = (a, b)$. Further, assume that the pair (a_i, b_i) is defined for some positive integer i. If $a_i = b_i$ then this is the last pair in the sequence. Otherwise we let $(a_{i+1}, b_{i+1}) = (a_i - b_i, b_i)$ if $a_i > b_i$ and $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ if $a_i < b_i$. Since $a, b > 0$, it follows by induction that $a_i, b_i > 0$ for all i. If the sequence

$(a_1, b_1), (a_2, b_2), \dots$ is finite and contains n pairs, then $a_n = b_n$ and we are done. If the sequence is infinite, it is enough to show that $a_n + b_n \to 0$ as $n \to \infty$. For any positive integer n let $c_n = \min(a_n, b_n)$. Since $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are nonincreasing sequences of positive numbers, so is the sequence $c_1, c_2, \dots$. By construction, $a_{i+1} + b_{i+1} = (a_i + b_i) - c_i$ for all i. It follows that the series $c_1 + c_2 + \cdots$ is convergent. In particular, $c_n \to 0$ as $n \to \infty$. Note that if $c_{i+1} < c_i$ for some i, then $c_i = \max(a_{i+1}, b_{i+1})$ so that $a_{i+1} + b_{i+1} = c_i + c_{i+1}$. This implies $a_n + b_n \to 0$ as $n \to \infty$. ∎

Lemma 3.5 *For any $a, b \in \mathbb{R}$ and $\epsilon > 0$ there exist $a_0, b_0 > 0$, $a_0 + b_0 < \epsilon$, such that $a \wedge b = a_0 \wedge b_0$ in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.*

Proof. Note that $c \wedge c = 0$ for all $c \in \mathbb{R}$. Therefore in the case $a \wedge b = 0$ it is enough to take $a_0 = b_0 = c$, where $0 < c < \epsilon/2$.

Now assume that $a \wedge b \neq 0$. Clearly, in this case a and b are nonzero. Since $(-a) \wedge (-b) = a \wedge b$ and $(-a) \wedge b = a \wedge (-b) = b \wedge a$ for all $a, b \in \mathbb{R}$, it is no loss to assume that a and b are positive. By Lemma 3.4, there exist pairs of positive numbers $(a_1, b_1) = (a, b)$, $(a_2, b_2), \dots, (a_n, b_n)$ such that $(a_{i+1}, b_{i+1}) = (a_i - b_i, b_i)$ or $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ for $1 \leq i \leq n-1$, and also $a_n + b_n < \epsilon$ or $a_n = b_n$. Since $(a' - b') \wedge b' = a' \wedge b' - b' \wedge b' = a' \wedge b'$ and $a' \wedge (b' - a') = a' \wedge b' - a' \wedge a' = a' \wedge b'$ for all $a', b' \in \mathbb{R}$, it follows by induction that $a_i \wedge b_i = a \wedge b$, $i = 1, 2, \dots, n$. Then $a_n \neq b_n$ as $a_n \wedge b_n = a \wedge b \neq 0$. Thus $a_n + b_n < \epsilon$. ∎

Lemma 3.6 *An element $\xi \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ is the scissors congruence invariant of some interval exchange transformation in $\mathcal{G}_I$ if and only if $\xi \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.*

Proof. As already mentioned before, the SAF invariant of a restricted rotation of type (a, b) is $a \wedge b$. By Lemma 2.1, any $f \in \mathcal{G}_I$ is a product of restricted rotations. Since Inv is a homomorphism of the group $\mathcal{G}_I$ due to Lemma 3.3, we obtain that $\mathrm{Inv}(f)$ is a finite sum of wedge products. Hence $\mathrm{Inv}(f) \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.

Let l denote the length of the interval I. By Lemma 3.5, for any $a, b \in \mathbb{R}$ one can find $a_0, b_0 > 0$, $a_0 + b_0 < l$, such that $a \wedge b = a_0 \wedge b_0$. By the choice of a_0 and b_0, the group $\mathcal{G}_I$ contains a restricted rotation of type (a_0, b_0). It follows that any wedge product in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$ is the SAF invariant of some interval exchange transformation in $\mathcal{G}_I$. Since Inv is a homomorphism of $\mathcal{G}_I$, any sum of wedge products is also the SAF invariant of some $f \in \mathcal{G}_I$.

Any $\xi \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$ is a linear combination of wedge products with rational coefficients. Since $r(a \wedge b) = (ra) \wedge b$ for all $a, b \in \mathbb{R}$ and $r \in \mathbb{Q}$, the element ξ can also be represented as a sum of wedge products. By the above, $\xi = \mathrm{Inv}(f)$ for some $f \in \mathcal{G}_I$. ∎

4 Commutator group

We begin this section with a technical lemma that will be used in the proof of the principal Lemma 4.2 below.

Lemma 4.1 *Suppose $L_1, L_2, \dots, L_k$ are positive numbers. Then there exist positive numbers $l_1, l_2, \dots, l_m$ linearly independent over $\mathbb{Q}$ such that each L_i is a linear combination of $l_1, l_2, \dots, l_m$ with nonnegative integer coefficients.*

Proof. The proof is by induction on the number k of the reals $L_1, L_2, \dots, L_k$. The case $k = 1$ is trivial. Now assume that $k > 1$ and the lemma holds for the numbers $L_1, L_2, \dots, L_{k-1}$. That is, there exist positive numbers $l_1, l_2, \dots, l_m$ linearly independent over $\mathbb{Q}$ such that each L_i, $1 \le i < k$ is a linear combination of $l_1, l_2, \dots, l_m$ with nonnegative integer coefficients. If the reals $l_1, \dots, l_m$ and L_k are linearly independent over $\mathbb{Q}$, then we are done. Otherwise L_k is a linear combination of $l_1, \dots, l_m$ with rational coefficients. Let us separate positive and negative terms in this linear combination: $L_k = a_1 l_{i_1} + \dots + a_s l_{i_s} - (b_1 l_{j_1} + \dots + b_p l_{j_p})$, where a_{i_t}, b_{j_t} are positive rationals and the indices $i_1, \dots, i_s, j_1, \dots, j_p$ are all distinct. It is possible that there is no negative term at all. Since $l_1, \dots, l_m$ and L_k are positive numbers, we can find positive rationals $r_1, \dots, r_s$ such that $r_1 + \dots + r_s = 1$ and $l'_{i_t} = a_t l_{i_t} - r_t(b_1 l_{j_1} + \dots + b_p l_{j_p})$ is positive for $1 \le t \le s$. Let $l'_i = l_i$ for any $1 \le i \le m$ different from $i_1, \dots, i_s$. Then $l'_1, \dots, l'_m$ are positive numbers linearly independent over $\mathbb{Q}$. By construction, $L_k = l'_{i_1} + \dots + l'_{i_s}$ and $l_{i_t} = a_t^{-1} l'_{i_t} + a_t^{-1} r_t (b_1 l'_{j_1} + \dots + b_p l'_{j_p})$ for $1 \le t \le s$. Therefore each of the numbers $l_1, \dots, l_m$ and L_k is a linear combination of $l'_1, \dots, l'_m$ with nonnegative rational coefficients. It follows that each of the numbers $L_1, L_2, \dots, L_k$ is also a linear combination of $l'_1, \dots, l'_m$ with nonnegative rational coefficients. Then there exists a positive integer N such that each L_i is a linear combination of $l'_1/N, \dots, l'_m/N$ with nonnegative integer coefficients. ∎

Let us call a product of restricted rotations *balanced* if for any $a, b > 0$ the number of factors of type (a, b) in this product matches the number of factors of type (b, a).

Lemma 4.2 *Any interval exchange transformation with zero SAF invariant can be represented as a balanced product of restricted rotations.*

Proof. Consider an arbitrary interval exchange transformation f of an interval I. If f is the identity, then for any restricted rotation h on I we have $f = hh^{-1}$, which is a balanced product of restricted rotations. Now assume f is not the identity. Let $I = I_1 \cup \ldots \cup I_k$ be a partition of I into subintervals such that the restriction of f to any I_i is a translation. Note that $k \ge 2$. Let $L_1, L_2, \dots, L_k$ be lengths of the intervals $I_1, I_2, \dots, I_k$. By Lemma 4.1, one can find positive numbers $l_1, l_2, \dots, l_m$ linearly independent over $\mathbb{Q}$ such that each L_i is a linear combination of $l_1, l_2, \dots, l_m$ with nonnegative integer coefficients. Then each I_i can be partitioned into smaller intervals with lengths in the set $\mathcal{L} = \{l_1, l_2, \ldots, l_m\}$. Clearly, the restriction of f to any of the smaller intervals is a translation, hence Lemma 2.1 applies here. We obtain that f can be represented as a product of restricted rotations, $f = f_1 f_2 \ldots f_n$, such that the type (a, b) of any factor satisfies $a, b \in \mathcal{L}$. For any $i, j \in \{1, 2, \ldots, m\}$ let s_{ij} denote the number of factors of type (l_i, l_j) in this product. Then

$$\mathrm{Inv}(f) = \sum\nolimits_{i=1}^{n} \mathrm{Inv}(f_n) = \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{m} s_{ij}(l_i \wedge l_j) = \sum\nolimits_{1 \le i < j \le m} (s_{ij} - s_{ji})(l_i \wedge l_j).$$

Since the numbers $l_1, \ldots, l_m$ are linearly independent over $\mathbb{Q}$, it follows from Lemma 3.1 that the wedge products $l_i \wedge l_j$, $1 \le i < j \le m$, are linearly independent in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$. Therefore $\mathrm{Inv}(f) = 0$ only if $s_{ij} = s_{ji}$ for all i, j, $i < j$. Then $s_{ij} = s_{ji}$ for all $i, j \in \{1, 2, \ldots, m\}$, which means that the product $f_1 f_2 \ldots f_n$ is balanced. ∎

The next lemma is an extension of Lemma 2.6 that will be used in the proofs of Lemmas 4.4 and 4.5 below.

Lemma 4.3 *Given $a, b, \epsilon > 0$, there exist $a_0, b_0 > 0$, $a_0 + b_0 < \epsilon$, such that any restricted rotation f of type (a, b) can be represented as $f = hg$, where h is a restricted rotation of type (a_0, b_0) and g is a product of interval swap maps.*

Proof. Consider an arbitrary restricted rotation f of type (a', b'), where $a' \ne b'$. If $a' > b'$ then Lemma 2.6 implies that $f = hg$, where h is a restricted rotation of type $(a' - b', b')$ and g is an interval swap map. In the case $a' < b'$, we observe that the inverse map f^{-1} is a restricted rotation of type (b', a'). The same Lemma 2.6 implies that $f^{-1} = \tilde{g}\tilde{h}$, where $\tilde{h}$ is a restricted rotation of type $(b' - a', a')$ and $\tilde{g}$ is an interval swap map. Note that $f = \tilde{h}^{-1}\tilde{g}^{-1} = \tilde{h}^{-1}\tilde{g}$ and $\tilde{h}^{-1}$ is a restricted rotation of type $(a', b' - a')$.

By Lemma 3.4, there exist pairs of positive numbers $(a_1, b_1), \ldots, (a_n, b_n)$ such that

- $(a_1, b_1) = (a, b)$,
- $(a_{i+1}, b_{i+1}) = (a_i - b_i, b_i)$ or $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ for $1 \le i \le n - 1$,
- $a_n + b_n < \epsilon$ or $a_n = b_n$.

Clearly, $a_i \ne b_i$ for $1 \le i < n$. By induction, it follows from the above that there exist interval exchange transformations $f_1 = f, f_2, \ldots, f_n$ and $g_2, \ldots, g_n$ such that f_i is a restricted rotation of type (a_i, b_i), g_i is an interval swap map, and $f_{i-1} = f_i g_i$ for $2 \le i \le n$. We have $f = f_n g$, where $g = g_n g_{n-1} \ldots g_2$ is a product of interval swap maps. If $a_n + b_n < \epsilon$ then we are done. Otherwise $a_n = b_n$ so that f_n itself is an interval swap map, hence f is a product of interval swap maps. In this case, take an arbitrary restricted rotation h of type (a_0, b_0), where $a_0 = b_0 < \epsilon/2$. Since h is also an interval swap map, we obtain $f = h(hf)$, where hf is a product of interval swap maps. ∎

Lemma 4.4 *Let f_1 and f_2 be restricted rotations of the same type. Then $f_1^{-1}f_2$ is a product of interval swap maps.*

Proof. The lemma has already been proved in one case. If the supports of f_1 and f_2 do not overlap then $f_1^{-1}f_2$ is the product of three interval swap maps due to Lemma 2.5. We are going to reduce the general case to this particular one.

Let (a, b) be the type of the restricted rotations f_1 and f_2. First we assume there exists an interval $I_0 \subset I$ of length $a+b$ that does not overlap with supports of f_1 and f_2. Let f_0 denote a unique restricted rotation of type (a, b) with support I_0. By Lemma 2.5, both $f_1^{-1}f_0$ and $f_0^{-1}f_2$ are products of three interval swap maps. Hence $f_1^{-1}f_2 = (f_1^{-1}f_0)(f_0^{-1}f_2)$ is the product of six interval swap maps.

The above assumption always holds in the case when $a + b \leq l/5$, where l is the length of I. Indeed, let us divide the interval I into 5 pieces of length $l/5$. Then the support of f_1, which is an interval of length $a+b$, overlaps with at most two pieces. The same is true for the support of f_2. Therefore we have at least one piece with interior disjoint from both supports. This piece clearly contains an interval of length $a + b$.

Now consider the general case. It follows from Lemma 4.3 that $f_1 = h_1g_1$ and $f_2 = h_2g_2$, where g_1, g_2 are products of interval swap maps while h_1, h_2 are restricted rotations of the same type (a_0, b_0) such that $a_0 + b_0 < l/5$. Note that g_1^{-1} and g_2^{-1} are also products of interval swap maps. By the above $h_1^{-1}h_2$ is the product of six interval swap maps. Then $f_1^{-1}f_2 = g_1^{-1}(h_1^{-1}h_2)g_2$ is a product of interval swap maps as well. ∎

Lemma 4.5 *Let f be a restricted rotation and g be an arbitrary interval exchange transformation. Then the commutator $f^{-1}g^{-1}fg$ is a product of interval swap maps.*

Proof. Let (a, b) be the type of the restricted rotation f and J be the support of f. First assume that the restriction of the transformation g^{-1} to J is a translation. Then $g^{-1}fg$ is also a restricted rotation of type (a, b), with support $g^{-1}(J)$. Therefore $f^{-1}g^{-1}fg$ is a product of interval swap maps due to Lemma 4.4.

In the general case, we choose an interval $I_0 \subset I$ such that g^{-1} is a translation when restricted to I_0. Let ϵ denote the length of I_0. According to Lemma 4.3, we have $f = f_0g_0$, where g_0 is a product of interval swap maps and f_0 is a restricted rotation of some type (a_0, b_0) such that $a_0 + b_0 < \epsilon$. Obviously, g_0^{-1} is also a product of interval swap maps. Since $a_0 + b_0 < \epsilon$, there exists a restricted rotation f_1 of type (a_0, b_0) with support contained in I_0. By the above the commutator $f_1^{-1}g^{-1}f_1g$ is a product of swap maps. By Lemma 4.4, $f_0^{-1}f_1$ and $f_1^{-1}f_0$ are also products of interval swap maps. Note that

$$f^{-1}g^{-1}fg = g_0^{-1}f_0^{-1}g^{-1}f_0g_0g = g_0^{-1}(f_0^{-1}f_1)(f_1^{-1}g^{-1}f_1g)g^{-1}(f_1^{-1}f_0)g_0g.$$

Therefore $f^{-1}g^{-1}fg = g_1g^{-1}g_2g$, where g_1 and g_2 are products of interval swap maps. Consider an arbitrary factorization $g_2 = h_1h_2 \dots h_n$ such that each h_i is an interval swap map. Then $g^{-1}g_2g = (g^{-1}h_1g)(g^{-1}h_2g)\dots(g^{-1}h_ng)$. Clearly, each $g^{-1}h_ig$ is an interval exchange transformation of order 2 and hence a product of interval swap maps due to Lemma 2.2. It follows that $f^{-1}g^{-1}fg$ can also be represented as a product of interval swap maps. ∎

Lemma 4.6 *Any balanced product of restricted rotations is also a product of interval swap maps.*

Proof. The proof is by strong induction on the number n of factors in a balanced product. Let $f = f_1f_2\dots f_n$ be a balanced product of n restricted rotations and assume that the lemma holds for any balanced product of less than n factors. Let (a,b) be the type of f_1. First consider the case $a = b$. In this case, f_1 is an interval swap map. If $n = 1$ then we are done. Otherwise $f = f_1g$, where $g = f_2\dots f_n$ is a balanced product of $n-1$ restricted rotations. By the inductive assumption, g is a product of interval swap maps, and so is f.

Now consider the case $a \neq b$. In this case, there is also a factor f_k of type (b,a). Let g_1 be the identity if $k = 2$ and $g_1 = f_2\dots f_{k-1}$ otherwise. Let g_2 be the identity if $k = n$ and $g_2 = f_{k+1}\dots f_n$ otherwise. We have

$$f = f_1g_1f_kg_2 = (f_1f_k)(f_k^{-1}g_1f_kg_1^{-1})(g_1g_2).$$

Since f_1^{-1} is a restricted rotation of type (b,a), it follows from Lemma 4.4 that $f_1f_k = (f_1^{-1})^{-1}f_k$ is a product of interval swap maps. Since $f_k^{-1}g_1f_kg_1^{-1}$ is the commutator of the restricted rotation f_k and the interval exchange transformation g_1^{-1}, it is a product of interval swap maps due to Lemma 4.5. If $n = 2$ then g_1g_2 is the identity and we are done. Otherwise we observe that g_1g_2 is a balanced product of $n-2$ restricted rotations. By the inductive assumption, g_1g_2 is a product of interval swap maps, and so is f. ∎

Proof of Theorem 1.1. Let $\mathcal{G} = \mathcal{G}_I$ be the group of interval exchange transformations of an arbitrary interval $I = [p,q)$. Let $\mathcal{G}_0$ be the set of all elements in $\mathcal{G}$ with zero SAF invariant. $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$ as it is the kernel of the homomorphism Inv (see Lemma 3.3). Let $\mathcal{G}_1$ denote the commutator group of $\mathcal{G}$, i.e., the subgroup of $\mathcal{G}$ generated by commutators $f^{-1}g^{-1}fg$, where $f,g \in \mathcal{G}$. Also, let $\mathcal{G}_2$ be the subgroup of $\mathcal{G}$ generated by all elements of order 2 and $\mathcal{G}_3$ be the subgroup generated by all elements of finite order. We have to prove that the groups $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ coincide.

Since the scissors congruence invariant Inv is a homomorphism of $\mathcal{G}$ to an abelian group, it vanishes on every commutator. It follows that $\mathcal{G}_1 \subset \mathcal{G}_0$. Lemmas 4.2 and 4.6 imply that any element of $\mathcal{G}_0$ is a product of interval swap maps, which are elements of order 2. Therefore $\mathcal{G}_0 \subset \mathcal{G}_2$. The inclusion $\mathcal{G}_2 \subset \mathcal{G}_3$ is trivial. By Lemma 2.2, any element of $\mathcal{G}_3$ is a product of interval swap maps, which are commutators due to Lemma 2.3. Hence $\mathcal{G}_3 \subset \mathcal{G}_1$. We conclude that $\mathcal{G}_0 = \mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3$. ∎

Proof of Theorem 1.2. According to Lemma 3.3, the SAF invariant Inv, regarded as a function on the group $\mathcal{G}_I$ of interval exchange transformations of an interval I, is a homomorphism to $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$. Therefore the quotient of $\mathcal{G}_I$ by the kernel of this homomorphism is isomorphic to its image. By Lemma 3.6, the image of the homomorphism is $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$. By Theorem 1.1, the kernel is the commutator group of $\mathcal{G}_I$. ∎

5 Simplicity

Let $\mathcal{G} = \mathcal{G}_I$ be the group of interval exchange transformations of an arbitrary interval $I = [p, q)$. In this section we show that the commutator group $[\mathcal{G}, \mathcal{G}]$ of $\mathcal{G}$ is simple.

Lemma 5.1 *For any $\epsilon > 0$ the commutator group of $\mathcal{G}$ is generated by interval swap maps of types less than ϵ.*

Proof. Let f be an arbitrary interval swap map in $\mathcal{G}$. Denote by a the type of f. Let $[x, x+a)$ and $[y, y+a)$ be the nonoverlapping intervals interchanged by f. We choose a sufficiently large positive integer N such that $a/N < \epsilon$. For any $i \in \{1, 2, \dots, N\}$ let f_i denote the interval exchange transformation that interchanges intervals $[x + (i-1)a/N, x + ia/N)$ and $[y + (i-1)a/N, y + ia/N)$ by translation while fixing the rest of the interval I. It is easy to see that $f = f_1 f_2 \dots f_N$. Note that each f_i is an interval swap map of type $a/N < \epsilon$.

Let H_ϵ be the subgroup of $\mathcal{G}$ generated by all interval swap maps of types less than ϵ. By the above the group H_ϵ contains all interval swap maps in $\mathcal{G}$. In view of Lemma 2.2, H_ϵ coincides with the subgroup of $\mathcal{G}$ generated by all elements of finite order. By Theorem 1.1, $H_\epsilon = [\mathcal{G}, \mathcal{G}]$. ∎

Lemma 5.2 *There exists $\epsilon > 0$ such that any two interval swap maps in $\mathcal{G}$ of the same type $a < \epsilon$ are conjugated in $[\mathcal{G}, \mathcal{G}]$.*

Proof. Let l be the length of the interval I. Consider arbitrary interval swap maps $f_1, f_2 \in \mathcal{G}$ of the same type $a < l/10$. Let us divide the interval I into 10 pieces of length $l/10$. The support of f_1 is the union of two intervals of length a. Since $a < l/10$, each interval of length a overlaps with at most two of the ten pieces. Hence the support of f_1 overlaps with at most 4 pieces. The same is true for the support of f_2. Therefore we have at least two pieces with interior disjoint from both supports. Clearly, one can find two nonoverlapping intervals I_1 and I_2 of length a in these pieces. Let f_0 be the interval swap map of type a that interchanges I_1 and I_2 by translation and fixes the rest of I. By construction, the support of f_0 does not overlap with the supports of f_1 and f_2. It follows from Lemma 2.4 that $f_1 = g_1 f_0 g_1$ and $f_0 = g_2 f_2 g_2$ for some elements $g_1, g_2 \in \mathcal{G}$ of order 2. By Theorem 1.1, the commutator group $[\mathcal{G}, \mathcal{G}]$ contains all elements of order

2 in $\mathcal{G}$. In particular, it contains f_1, f_2, g_1, and g_2. Then $g_2g_1 \in [\mathcal{G},\mathcal{G}]$ as well. Since $f_1 = g_1(g_2f_2g_2)g_1 = (g_2g_1)^{-1}f_2(g_2g_1)$, the elements f_1 and f_2 are conjugated in $[\mathcal{G},\mathcal{G}]$. ∎

Proof of Theorem 1.3. Suppose H is a nontrivial normal subgroup of $[\mathcal{G},\mathcal{G}]$. Let f be an arbitrary element of H different from the identity. By Lemma 2.7, there exist $\epsilon_1 > 0$ and, for any $0 < \epsilon < \epsilon_1$, interval swap maps $g_1, g_2 \in \mathcal{G}$ such that $g_2f^{-1}g_1fg_1g_2$ is an interval swap map of type ϵ. The interval swap maps g_1 and g_2 are involutions. They belong to $[\mathcal{G},\mathcal{G}]$ due to Lemma 2.3. Since H is a normal subgroup of $[\mathcal{G},\mathcal{G}]$ that contains f, it also contains the interval exchange transformations f^{-1}, $g_1^{-1}fg_1 = g_1fg_1$, $f^{-1}g_1fg_1$, and $g_2^{-1}(f^{-1}g_1fg_1)g_2 = g_2f^{-1}g_1fg_1g_2$. We obtain that for any $0 < \epsilon < \epsilon_1$ the subgroup H contains an interval swap map of type ϵ. By Lemma 5.2, there exists $\epsilon_2 > 0$ such that any two interval swap maps in $\mathcal{G}$ of the same type $\epsilon < \epsilon_2$ are conjugated in $[\mathcal{G},\mathcal{G}]$. It follows that all interval swap maps in $\mathcal{G}$ of types less than $\min(\epsilon_1, \epsilon_2)$ are also in H. According to Lemma 5.1, the commutator group of $\mathcal{G}$ is generated by these maps. Hence $H = [\mathcal{G},\mathcal{G}]$. Thus the only nontrivial normal subgroup of $[\mathcal{G},\mathcal{G}]$ is $[\mathcal{G},\mathcal{G}]$ itself. That is, $[\mathcal{G},\mathcal{G}]$ is a simple group. ∎

References

[1] P. Arnoux, Échanges d'intervalles et flots sur les surfaces. In *Théorie Ergodique*, 5–38. Monograph. Enseign. Math. **29**, Univ. Genève, Geneva, 1981 (in French).

[2] A. B. Katok, A. M. Stepin, Approximations in ergodic theory. *Uspehi Mat. Nauk* **22** (1967), no. 5(137), 81–106 (in Russian).

[3] M. Keane, Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.

[4] Chih-Han Sah, Scissors congruences of the interval. Preprint, 1981.

[5] W. A. Veech, The metric theory of interval exchange transformations III. The Sah-Arnoux-Fathi invariant. *Amer. J. Math.* **106** (1984), no. 6, 1389–1422.

[6] M. Viana, Ergodic theory of interval exchange maps. *Rev. Mat. Complut.* **19** (2006), no. 1, 7–100.