On a substitution subshift related to the Grigorchuk group

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1 Introduction

Let τ be a substitution over the alphabet $\{a, b, c, d\}$ defined by relations

$$\tau(a) = aca, \quad \tau(b) = d, \quad \tau(c) = b, \quad \tau(d) = c.$$

The substitution acts on words (finite sequences) over this alphabet as well as on infinite sequences. It is easy to observe that τ has a unique invariant sequence ω that is the limit of words $\tau^k(a)$, $k = 1, 2, \dots$. Let Ω be the smallest closed set of one-sided infinite sequences over the alphabet $\{a, b, c, d\}$ that contains ω and is invariant under the shift σ (σ acts on sequences by deleting the first element). The set Ω consists of those sequences for which any finite subword appears somewhere in ω. The restriction of σ to Ω is called a subshift. Since ω is a fixed point of a substitution, this particular subshift is called a *substitution subshift*.

The substitution τ plays an important role in the study of the Grigorhuk group (see the survey [3]). The Grigorchuk group G is a finitely generated infinite group where all elements are of finite order. It has many other remarkable properties as well. The group has four generators a, b, c, d. An important fact is that the substitution τ gives rise to a homomorphism of G to itself. It follows that τ transforms any relator for G into another relator. Although G has no finite presentation, it admits a recursive presentation (due to Lysenok [4]) obtained from a finite set of relators by repeatedly applying τ:

$$G = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \tau^n((ad)^4) = \tau^n((adacac)^4),\ n \geq 0 \rangle.$$

The structure of the Grigorchuk group is not completely understood yet. In view of the above presentation, it is believed that properties of the sequence ω and the subshift $\sigma|_\Omega$ might give an insight on that matter. In this paper we study dynamics of $\sigma|_\Omega$.

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Theorem 1.1 *The subshift $\sigma|_\Omega$ is, up to a countable set, continuously conjugated to the binary odometer.*

Theorem 1.1 suggests that ergodic properties of the subshift $\sigma|_\Omega$ are the same as ergodic properties of the binary odometer. The following theorem is a detailed version of this suggestion.

Theorem 1.2 *(i) The subshift $\sigma|_\Omega$ has a unique invariant Borel probability measure μ.*

(ii) $\sigma|_\Omega$ is ergodic with respect to the measure μ. Moreover, every orbit of $\sigma|_\Omega$ is uniformly distributed in Ω with respect to μ.

(iii) $\sigma|_\Omega$ has purely point spectrum, the eigenvalues being all roots of unity of order $1, 2, \dots, 2^k, \dots$. Each eigenvalue is simple.

(iv) All eigenfunctions of $\sigma|_\Omega$ are continuous.

It turns out that ω is a one-sided analog of what is called Toeplitz sequences (see, e.g., [2]). The Toeplitz sequences can be informally described as almost periodic. Subshifts generated by Toeplitz sequences are known to be continuous extensions of odometers. A nontrivial feature of ω is that the extension is one-to-one up to a countable set. In Section 4 we place ω into a class of Toeplitz sequences that are as close to periodic as possible. Theorems 1.1 and 1.2 hold for all sequences in that class.

The substitution subshifts can be studied by means of Bratteli diagrams (see, e.g., [1]). This approach works very well for so-called primitive substitutions. However the substitution τ is not primitive.

The paper is organized as follows. Section 2 is a survey on odometers. We discuss their dynamics and determine when a particular odometer is a continuous factor of a particular topological dynamical system. In Section 3 we collect necessary information about Toeplitz sequences and the associated subshifts. In Section 4, these results are applied to the class of Toeplitz sequences that contains ω. The paper ends with the proof of Theorems 1.1 and 1.2. It should be noted that a number of results in Sections 2 and 3 are well known to specialists (although it might not be easy to locate them in the literature; in particular, results on Toelitz sequences are usually formulated for bi-infinite sequences). For reader's convenience, we include all proofs so that the paper is self-contained.

2 Odometers

In this section we consider general transformations $T : X \to X$ such that X is a compact topological space and T is a continuous map (not necessarily one-to-one or onto). For any $x \in X$ the sequence $x, Tx, T^2x, \dots$ is called the *orbit* of the point x under the transformation T. By $Z(x, T)$ we denote the closure of the orbit. Note that $Z(x, T)$ is the smallest closed subset of X that contains x and is invariant under T. If C is a compact subset of X invariant under T, then $T|_C$ denotes the restriction of T to C. The transformation T is called *transitive* if there exists a dense orbit, that is, if $Z(x,T) = X$ for some $x \in X$. T is called *minimal* if each orbit is dense.

Let $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ be continuous transformations of compact sets. Suppose there exists a continuous map $f : X_1 \to X_2$ such that f is onto and $fT_1 = T_2 f$ so that the following diagram is commutative:

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "f"'] & X_1 \arrow[d, "f"] \\ X_2 \arrow[r, "T_2"] & X_2 \end{tikzcd}$$

Then T_2 is called a *(continuous) factor* of T_1 while T_1 is called a *(continuous) extension* of T_2. If, in addition, we can choose f to be a homeomorphism then T_1 and T_2 are called *(continuously) conjugated* and f is called a *conjugacy*.

A transformation $T : X \to X$ is called a *cyclic permutation* if X is a finite set with the discrete topology, T is one-to-one, and an orbit of T contains all elements of X. Let n denote the cardinality of X. Then for any $x \in X$ the sequence $x, Tx, T^2x, \dots, T^{n-1}x$ is a complete list of elements of X. Besides, $T^n x = x$ so that T has order n. Any cyclic permutation is determined by its order up to conjugacy.

It is easy to observe that a cyclic permutation T_1 is a factor of another cyclic permutation T_2 if and only if the order of T_1 divides the order of T_2. In general, a cyclic permutation of order n is a factor of a transformation $T : X \to X$ if and only if the set X can be split into n disjoint closed subsets $X_1, X_2, \dots, X_n$ which are cyclically permuted by T, that is, $T(X_i) \subset X_{i+1}$ for $1 \le i \le n-1$ and $T(X_n) \subset X_1$. Note that the sets $X_1, X_2, \dots, X_n$ are both closed and open.

Now assume that a transformation $T : X \to X$ is a continuous extension of two cyclic permutations T_1 and T_2 such that T_1 is a factor of T_2. Then there exist continuous onto maps $f : X_2 \to X_1$, $f_1 : X \to X_1$, and $f_2 : X \to X_2$ such that $fT_2 = T_1 f$, $f_1 T = T_1 f_1$, and $f_2 T = T_2 f_2$, i.e., the following diagrams are commutative:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f_2"'] & X \arrow[d, "f_2"] \\ X_2 \arrow[r, "T_2"] \arrow[d, "f"'] & X_2 \arrow[d, "f"] \\ X_1 \arrow[r, "T_1"] & X_1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f_1"'] & X \arrow[d, "f_1"] \\ X_1 \arrow[r, "T_1"] & X_1 \end{tikzcd}$$

Note that, given f and f_2, we can always take $f_1 = ff_2$. It turns out that the latter identity can also be satisfied when one is given f and f_1 and has to choose f_2.

Lemma 2.1 *For any choice of the maps f and f_1 above, we can choose the map f_2 so that $ff_2 = f_1$.*

Proof. Take an arbitrary continuous map $h : X \to X_2$ such that $hT = T_2h$. For any $x \in X_1$ and $y \in X_2$ let $U(x) = f_1^{-1}(x)$, $V(x) = h^{-1}(y)$, and $W(x,y) = U(x) \cap V(y)$. Then $U(x)$, $x \in X_1$ is a collection of disjoint closed sets that partition X. The same holds true for the collections $V(y)$, $y \in X_2$ and $W(x,y)$, $(x,y) \in X_1 \times X_2$. Furthermore, $T(U(x)) \subset U(T_1x)$, $T(V(y)) \subset V(T_2y)$, and $T(W(x,y)) \subset W(T_1x, T_2y)$.

By n denote the cardinality of the set X_1. For any $y \in X_2$ let $Y(y) = \bigcup_{k=0}^{n-1} W(f(y), T_2^k y)$. Clearly, each $Y(y)$ is a closed subset of $U(f(y))$. Besides, $T(Y(y)) \subset Y(T_2y)$ since $T_1f(y) = f(T_2y)$. For any $x_0 \in X_1$ and $y_0 \in X_2$ there is a unique $k \in \{0, 1, \ldots, n-1\}$ such that $f(T_2^{-k}y_0) = x_0$. It follows that the sets $Y(y)$, $y \in X_2$ are disjoint and cover the entire set X.

Define a map $f_2 : X \to X_2$ so that $f_2(z) = y$ for all $y \in X_2$ and $z \in Y(y)$. Since $Y(y)$, $y \in X_2$ is a collection of disjoint closed sets that partition X, the map f_2 is well defined and continuous. Since $Y(y) \subset U(f(y))$ and $T(Y(y)) \subset Y(T_2y)$ for any $y \in X_2$, it follows that $ff_2 = f_1$ and $f_2T = T_2f_2$. Clearly, f_2 is onto. ■

For any integer $n > 0$ we denote $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Given $k \in \mathbb{Z}_n$ and $m \in \mathbb{Z}$, the sum $k + m$ is a well defined element of $\mathbb{Z}_n$. The *odometer* on $\mathbb{Z}_n$ is the transformation $x \mapsto x + 1$. Now let $n_1, n_2, \ldots$ be a finite or infinite sequence of positive integers. The *odometer* on $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots$ is a transformation T defined as follows. For any $m_i \in \mathbb{Z}_{n_i}$, $i = 1, 2, \ldots$, we let $T(m_1, m_2, \ldots) = (k_1, k_2, \ldots)$, where $k_i = m_i + 1$ if $m_j = -1 + n_j\mathbb{Z}$ for $1 \leq j < i$, and $k_i = m_i$ otherwise. We regard each $\mathbb{Z}_n$ as a discrete topological space and endow $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots$ with the product topology. Then the odometer is a homeomorphism of a compact set. It is easy to see that any odometer is minimal. The odometer on a finite set is a cyclic permutation.

Lemma 2.2 *For any odometer T_0 there exist a compact Abelian group G and $g_0 \in G$ such that T_0 is continuously conjugated to the transformation $g \mapsto g + g_0$ of G.*

Proof. Suppose T_0 is the odometer on $X = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots$. If X is finite then T_0 is a cyclic permutation, hence it is conjugated to the odometer on some $\mathbb{Z}_n$. Now assume X is infinite. Let us regard $\mathbb{Z}$ as a discrete topological group and endow the countable product $G = \mathbb{Z} \times \mathbb{Z} \times \ldots$ with the product topology. By G_0 denote the closed subgroup of G generated by elements $(n_1, -1, 0, 0, 0, \ldots)$, $(0, n_2, -1, 0, 0, \ldots)$, $(0, 0, n_3, -1, 0, \ldots)$, ... Let $g_0 \in G/G_0$ be the coset containing $(1, 0, 0, \ldots)$. Each $g \in G/G_0$ intersects the compact set $Y = \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\} \times \ldots$ in exactly one element. For any $y = (y_1, y_2, \ldots) \in Y$ let $f_1(y) = (y_1 + n_1\mathbb{Z}, y_2 + n_2\mathbb{Z}, \ldots) \in X$ and let $f_2(y) \in G/G_0$ be the coset containing y. It is easy to observe that the maps $f_1 : Y \to X$ and

$f_2 : Y \to G/G_0$ are homeomorphisms. Besides, $f_2 f_1^{-1}(T_0 x) = f_2 f_1^{-1}(x) + g_0$ for all $x \in X$. ∎

Lemma 2.3 *Assume that a transformation $T : X \to X$ is a continuous extension of an odometer $T_0 : X_0 \to X_0$. Then for any $x \in X$ and $x_0 \in X_0$ there exists a continuous map $f : X \to X_0$ such that f is onto, $fT = T_0 f$, and $f(x) = x_0$. The map f is unique provided that T is transitive.*

Proof. It is no loss of generality to replace T_0 by a continuously conjugated transformation. In view of Lemma 2.2, we can assume that X_0 is a compact Abelian group and $T_0 x_0 = x_0 + g_0$ for some $g_0 \in X_0$ and all $x_0 \in X_0$. Let $f : X \to X_0$ be a continuous onto map such that $fT = T_0 f$. For any $g \in X_0$ and $x \in X$ let $f_g(x) = f(x) + g$. Then f_g is a continuous map of X onto X_0 and $f_g T(x) = fT(x) + g = T_0 f(x) + g = f(x) + g_0 + g = T_0 f_g(x)$ for all $x \in X$. Obviously, for any $x \in X$ and $x_0 \in X_0$ there exists a unique $g \in X_0$ such that $f_g(x) = x_0$.

Now assume T is transitive and pick $y \in X$ such that the orbit of y under the transformation T is dense in X. Suppose $h : X \to X_0$ is a continuous map such that $hT = T_0 h$. We have $h(y) = f_g(y)$ for some $g \in X_0$. Since $hT = T_0 h$ and $f_g T = T_0 f_g$, it follows that $h(T^i y) = f_g(T^i y)$ for $i = 1, 2, \dots$. Then density of the sequence $y, Ty, T^2 y, \dots$ in X implies that $h = f_g$. ∎

Lemma 2.4 *Two odometers are continuously conjugated if either of them is a continuous factor of the other.*

Proof. Let $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ be odometers such that T_1 is both a continuous factor and a continuous extension of T_2. Pick $x_1 \in X_1$ and $x_2 \in X_2$. By Lemma 2.3, there are unique continuous onto maps $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$ such that $f_1 T_1 = T_2 f_1$, $f_2 T_2 = T_1 f_2$, $f_1(x_1) = x_2$, and $f_2(x_2) = x_1$. Note that $f_1 f_2 f_1$ and $f_2 f_1 f_2$ are continuous maps, $f_1 f_2 f_1(X_1) = X_2$, and $f_2 f_1 f_2(X_2) = X_1$. Further, $(f_1 f_2 f_1) T_1 = f_1 f_2 T_2 f_1 = f_1 T_1 f_2 f_1 = T_2 (f_1 f_2 f_1)$ and $f_1 f_2 f_1(x_1) = x_2$. Similarly, $(f_2 f_1 f_2) T_2 = T_1 (f_2 f_1 f_2)$ and $f_2 f_1 f_2(x_2) = x_1$. It follows that $f_1 f_2 f_1 = f_1$ and $f_2 f_1 f_2 = f_2$. Since the maps f_1 and f_2 are onto, $f_1 f_2$ and $f_2 f_1$ are the identity maps of X_2 and X_1, respectively. Thus f_1 and f_2 are homeomorphisms. ∎

The following two lemmas explore relations between odometers and cyclic permutations.

Lemma 2.5 *Assume that a transformation T is a continuous extension of cyclic permutations of orders $n_1, n_1 n_2, n_1 n_2 n_3, \dots$, where $n_1, n_2, n_3, \dots$ is a sequence of positive integers. Then T is also a continuous extension of the odometer on $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \dots$.*

Proof. We assume that the sequence $n_1, n_2, n_3, \dots$ is infinite as otherwise the lemma is trivial. For any $k \geq 1$ let $X_k = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and denote by T_k the odometer on X_k. Also, let T_∞ denote the odometer on $X_\infty = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \dots$ and let X denote the space on which T acts. For any $k \geq 1$ consider the natural projections $\pi_k : X_{k+1} \to X_k$ and $p_k : X_\infty \to X_k$. They are continuous and onto. Besides, $\pi_k T_{k+1} = T_k \pi_k$ and $p_k T_\infty = T_k p_k$. Since the odometer T_k is a cyclic permutation of order $n_1 n_2 \cdots n_k$, it is a factor of the transformation T. Hence there is a continuous map $f_k : X \to X_k$ such that $f_k T = T_k f_k$. In view of Lemma 2.1, we can choose the maps $f_1, f_2, \dots$ so that $f_k = \pi_k f_{k+1}$ for all $k \geq 1$.

Define a map $f : X \to X_\infty$ as follows. Given $x \in X$, let $f(x) = (m_1, m_2, \dots)$, where $f_k(x) = (m_1, m_2, \dots, m_k)$ for all $k \geq 1$. The map f is well defined since $f_k = \pi_k f_{k+1}$ for all $k \geq 1$. Its continuity follows from the continuity of $f_1, f_2, \dots$. Furthermore, $fT = T_\infty f$ as $f_k T = T_k f_k$ and $p_k T_\infty = T_k p_k$ for all $k \geq 1$. In particular, the image $f(X)$ is invariant under T_∞. Since X is compact, $f(X)$ a nonempty compact subset of X_∞. The minimality of the odometer T_∞ implies that the map f is onto. Thus T_∞ is a continuous factor of T. ∎

Lemma 2.6 *A cyclic permutation of order n is a continuous factor of the odometer on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \times \dots$ if and only if n divides some of the numbers $m_1, m_1 m_2, m_1 m_2 m_3, \dots$.*

Proof. Let T be the odometer on $X = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots$. First suppose the sequence $m_1, m_2, \dots$ is finite. Denote by k its length. Then T is a cyclic permutation of order $m_1 m_2 \cdots m_k$. Hence a cyclic permutation of order n is a factor of T if and only if n divides $m_1 m_2 \cdots m_k$.

Now consider the case when the sequence $m_1, m_2, \dots$ is infinite. Suppose that a cyclic permutation $T_0 : X_0 \to X_0$ of order n is a factor of T. Let $f : X \to X_0$ be a continuous map such that $fT = T_0 f$. It is easy to observe that for any $x \in X$ the sequence $T^{m_1}x, T^{m_1 m_2}x, T^{m_1 m_2 m_3}x, \dots$ converges to x. Hence the sequence $f(T^{m_1}x), f(T^{m_1 m_2}x), f(T^{m_1 m_2 m_3}x), \dots$ converges to $f(x)$. Since X_0 is a finite set with the discrete topology, this means that $f(T^{m_1 m_2 \cdots m_k}x) = T_0^{m_1 m_2 \cdots m_k} f(x)$ coincides with $f(x)$ for large k. It follows that n divides $m_1 m_2 \cdots m_k$ for large k.

Conversely, if n divides some $m_1 m_2 \cdots m_k$ then any cyclic permutation T_0 of order n is a factor of the odometer T_k on $X_k = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$. But T_k is a factor of T since the natural projection $p_k : X \to X_k$ is continuous and satisfies $p_k T = T_k p_k$. Then T_0 is also a factor of the odometer T. ∎

Lemma 2.7 *The odometer on $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \dots$ is a continuous factor of the odometer on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \times \dots$ if and only if each element of the sequence $n_1, n_1 n_2, n_1 n_2 n_3, \dots$ divides an element of the sequence $m_1, m_1 m_2, m_1 m_2 m_3, \dots$.*

Proof. Let T_1 denote the odometer on $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \dots$ and T_2 denote the odometer on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \times \dots$. By Lemma 2.6, cyclic permutations of orders

$n_1, n_1n_2, n_1n_2n_3, \dots$ are factors of T_1. Assume that T_1 is a factor of T_2. Then all factors of T_1 are also factors of T_2. It follows from Lemma 2.6 that each of the numbers $n_1, n_1n_2, n_1n_2n_3, \dots$ divides some of the numbers $m_1, m_1m_2, m_1m_2m_3, \dots$.

Conversely, assume that each of the numbers $n_1, n_1n_2, n_1n_2n_3, \dots$ divides some of the numbers $m_1, m_1m_2, m_1m_2m_3, \dots$. Then Lemma 2.6 implies that cyclic permutations of orders $n_1, n_1n_2, n_1n_2n_3, \dots$ are factors of the odometer T_2. By Lemma 2.5, the odometer T_1 is a factor of T_2 as well. ∎

To any continuous transformation T of a compact topological space we associate the set $\mathrm{CF}(T)$ of positive integers n such that T is a continuous extension of the cyclic permutation of order n (CF stands for "cyclic factors").

Lemma 2.8 *The set* $\mathrm{CF}(T)$ *has the following properties:*

(i) $1 \in \mathrm{CF}(T)$*;*

(ii) if $n \in \mathrm{CF}(T)$ *and* $d > 0$ *is a divisor of* n*, then* $d \in \mathrm{CF}(T)$*;*

(iii) if $n_1, n_2, \dots, n_k \in \mathrm{CF}(T)$*, then* $\mathrm{lcm}(n_1, n_2, \dots, n_k) \in \mathrm{CF}(T)$.

Proof. Property (i) is trivial.

Suppose d and n are positive integers. If d divides n then a cyclic permutation of order d is a factor of a cyclic permutation of order n. Therefore all continuous extensions of the latter permutation are also continuous extensions of the former one. In particular, $d \in \mathrm{CF}(T)$ whenever $n \in \mathrm{CF}(T)$. Property (ii) is verified.

Let X denote the topological space on which T acts. Given $m, n \in \mathrm{CF}(T)$, there exist continuous maps $f_1 : X \to \mathbb{Z}_m$ and $f_2 : X \to \mathbb{Z}_n$ such that $f_1(Tx) = f_1(x) + 1$ and $f_2(Tx) = f_2(x) + 1$ for all $x \in X$. Then $f = (f_1, f_2)$ is a continuous map of X to $\mathbb{Z}_m \times \mathbb{Z}_n$. Furthermore, $fT = T_0 f$, where T_0 denotes the transformation $(x, y) \mapsto (x + 1, y + 1)$ of $\mathbb{Z}_m \times \mathbb{Z}_n$. Assume that m and n are coprime. Then T_0 is a cyclic permutation of order mn and the map f is onto. Consequently, $mn \in \mathrm{CF}(T)$.

Let $n_1, n_2 \in \mathrm{CF}(T)$. It is easy to show that there exist positive coprime integers d_1, d_2 such that d_1 divides n_1, d_2 divides n_2, and $d_1d_2 = \mathrm{lcm}(n_1, n_2)$. By the above $d_1, d_2, d_1d_2 \in \mathrm{CF}(T)$. Now property (iii) follows by induction. ∎

The continuous cyclic factors of a transformation $T : X \to X$ are related to continuous eigenfunctions of T. Let $\mathcal{U}_T$ denote a linear operator that acts on functions on X by precomposing them with T: $\mathcal{U}_T\phi = \phi T$. A nonzero function $\phi : X \to \mathbb{C}$ is an *eigenfunction* of T associated with an *eigenvalue* λ if $\mathcal{U}_T\phi = \lambda\phi$, that is, if $\phi(Tx) = \lambda\phi(x)$ for all $x \in X$.

Lemma 2.9 *If* $n \in \mathrm{CF}(T)$*, then the transformation* T *admits a continuous eigenfunction associated with the eigenvalue* $e^{2\pi i/n}$*, a primitive* n*th root of unity. For a transitive* T*, the converse is true as well. Besides, for a transitive* T *any continuous eigenfuction is determined by its eigenvalue uniquely up to scaling.*

Proof. Given a positive integer n, let R_n denote the set of all nth roots of unity in $\mathbb{C}$. R_n is a multiplicative cyclic group of order n generated by a primitive root $\zeta_n = e^{2\pi i/n}$. A transformation $T_n : R_n \to R_n$ defined by $T_n(z) = \zeta_n z$ is a cyclic permutation of order n. If $n \in \mathrm{CF}(T)$ for some transformation $T : X \to X$, then there exists a continuous mapping $f : X \to R_n$ such that $fT = T_n f$, i.e., $f(Tx) = T_n(f(x)) = \zeta_n f(x)$ for all $x \in X$. Clearly, f is a continuous eigenfunction of T associated with the eigenvalue ζ_n.

Now assume that $T : X \to X$ is transitive and pick a point $x_0 \in X$ with dense orbit. Let ϕ be a continuous eigenfunction of T associated with an eigenvalue λ. The function ϕ is uniquely determined by its values on the orbit $x_0, Tx_0, T^2x_0, \dots$. Since $\phi(T^k x_0) = \lambda^k \phi(x_0)$ for $k = 1, 2, \dots$, the function ϕ is uniquely determined by $\phi(x_0)$ and λ. Note that $\phi(x_0) \neq 0$ as otherwise ϕ will be identically zero. Since nonzero scalar multiples of ϕ are also eigenfunctions with the same eigenvalue, ϕ is determined by the eigenvalue λ up to scaling.

In the case $\lambda = \zeta_n$, replace the eigenfunction ϕ by a scalar multiple so that $\phi(x_0) = 1$. Then $\phi(T^k x_0) \in R_n$ for all $k \geq 1$, which implies that $\phi(x) \in R_n$ for all $x \in X$. Therefore ϕ maps X onto R_n and $\phi(Tx) = \zeta_n \phi(x) = T_n(\phi(x))$ for all $x \in X$. Thus the cyclic permutation T_n is a factor of T, that is, $n \in \mathrm{CF}(T)$. ∎

Lemma 2.10 *Suppose F_0 is a set of positive integers such that (i) $1 \in F_0$, (ii) any positive divisor of an element of F_0 also belongs to F_0, and (iii) the least common multiple of finitely many elements of F_0 is in F_0 as well. Then there exists an odometer T_0 such that $\mathrm{CF}(T_0) = F_0$.*

Proof. First suppose that the set F_0 is finite. Let N be its maximal element. For any $m \in F_0$ the least common multiple of m and N belongs to F_0. By the choice of N, we have $\mathrm{lcm}(m, N) = N$, that is, m divides N. On the other hand, F_0 contains all positive divisors of N. Therefore F_0 is the set of positive integers that divide N. It follows that $F_0 = \mathrm{CF}(T_0)$ for any cyclic permutation T_0 of order N. One example of such a permutation is the odometer on $\mathbb{Z}_N$.

Now suppose that the set F_0 is infinite and let $n_1, n_2, \dots$ be a complete list of its elements. For any $k \geq 1$ let m_k be the least common multiple of the integers $n_1, n_2, \dots, n_k$. Then the numbers $m_1, m_2, \dots$ belong to F_0, each m_k divides m_{k+1}, and any $n \in F_0$ divides some m_k. On the other hand, all positive divisors of any m_k are in F_0. Hence F_0 is the set of positive integers that divide some of the numbers $m_1, m_2, \dots$. It follows from Lemma 2.6 that $F_0 = \mathrm{CF}(T_0)$, where T_0 is the odometer on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2/m_1} \times \mathbb{Z}_{m_3/m_2} \times \dots$. ∎

Proposition 2.11 *Suppose T_1 and T_2 are odometers. Then*

(i) T_1 is a continuous factor of T_2 if and only if $\mathrm{CF}(T_1) \subset \mathrm{CF}(T_2)$;

(ii) T_1 and T_2 are continuously conjugated if and only if $\mathrm{CF}(T_1) = \mathrm{CF}(T_2)$.

Proof. Let T_1 be the odometer on $\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\mathbb{Z}_{n_3}\times\dots$ and T_2 be the odometer on $\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\times\mathbb{Z}_{m_3}\times\dots$. Lemma 2.6 implies that $\mathrm{CF}(T_1)$ is the set of positive integers that divide some of the numbers $n_1, n_1n_2, n_1n_2n_3, \dots$. Likewise, $\mathrm{CF}(T_2)$ is the set of positive integers that divide some of the numbers $m_1, m_1m_2, m_1m_2m_3, \dots$. Therefore $\mathrm{CF}(T_1)\subset\mathrm{CF}(T_2)$ if and only if each of the numbers $n_1, n_1n_2, n_1n_2n_3, \dots$ divides some of $m_1, m_1m_2, m_1m_2m_3, \dots$. According to Lemma 2.7, this is exactly when T_1 is a factor of T_2.

By Lemma 2.4, the odometers T_1 and T_2 are conjugated if either of them is a factor of the other. Hence the statement (ii) of the proposition follows from the statement (i). ∎

Suppose that an odometer T_0 is a continuous factor of a transformation T. We shall say that T_0 is a *maximal odometer factor* of T if any odometer is a continuous factor of T_0 whenever it is a continuous factor of T.

Proposition 2.12 *(i) An odometer T_0 is a continuous factor of a transformation T if and only if $\mathrm{CF}(T_0)\subset\mathrm{CF}(T)$.*

(ii) An odometer T_0 is a maximal odometer factor of T if and only if $\mathrm{CF}(T_0)=\mathrm{CF}(T)$.

(iii) The maximal odometer factor always exists and is unique up to continuous conjugacy.

Proof. Let T_0 be the odometer on $\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\mathbb{Z}_{n_3}\times\dots$. By Lemma 2.6, the integers $n_1, n_1n_2, n_1n_2n_3, \dots$ belong to $\mathrm{CF}(T_0)$. If $\mathrm{CF}(T_0)\subset\mathrm{CF}(T)$, they belong to $\mathrm{CF}(T)$ as well. Then it follows from Lemma 2.5 that T_0 is a factor of T. Conversely, if T_0 is a factor of T then all factors of T_0 are also factors of T; in particular, $\mathrm{CF}(T_0)\subset\mathrm{CF}(T)$.

Lemmas 2.8 and 2.10 imply that for any transformation T there exists an odometer T_1 such that $\mathrm{CF}(T_1)=\mathrm{CF}(T)$. By the above T_1 is a factor of T. Suppose T_2 is another odometer that is a factor of T. Then $\mathrm{CF}(T_2)\subset\mathrm{CF}(T)=\mathrm{CF}(T_1)$. By Proposition 2.11, T_2 is a factor of T_1. Therefore T_1 is a maximal odometer factor of T. Its uniqueness up to continuous conjugacy follows from Lemma 2.4. Thus an odometer T_0 is a maximal odometer factor of T if and only if it is conjugated to T_1. According to Proposition 2.11, this is exactly when $\mathrm{CF}(T_0)=\mathrm{CF}(T_1)=\mathrm{CF}(T)$. ∎

Now let us consider ergodic properties of odometers. First recall some definitions. Let X be a measured space and $T:X\to X$ be a measurable transformation. A measure μ on X is *invariant* under T if $\mu(T^{-1}(B))=\mu(B)$ for any measurable subset $B\subset X$. Let $\mathcal{U}_T$ be the linear operator acting on functions on X by precomposition with T: $\mathcal{U}_T\phi=\phi T$. If μ is an invariant measure, then $\mathcal{U}_T$ is a unitary operator when restricted to the Hilbert space $L_2(X,\mu)$. The spectral properties of this unitary operator are referred to as *spectral properties* of the dynamical system (X,μ,T). For instance, one says that T has *pure point spectrum* if $L_2(X,\mu)$ admits an orthonormal basis consisting of eigenfunctions of $\mathcal{U}_T$.

The transformation T is *ergodic* with respect to the invariant measure μ if for any measurable subset $B \subset X$ that is backward invariant under T (i.e., $T^{-1}(B) \subset B$) one has $\mu(B) = 0$ or $\mu(X \setminus B) = 0$. If T is ergodic and the measure μ is finite, then Birkhoff's ergodic theorem implies that μ-almost all orbits of T are uniformly distributed in X relative to this measure.

A homeomorphism T of a compact topological space X is called *uniquely ergodic* if there exists a unique Borel probability measure on X invariant under T. The uniquely ergodic transformation T is ergodic with respect to the unique invariant measure. Moreover, in this case every orbit of T is uniformly distributed in X.

Proposition 2.13 *Let T_0 be an odometer. Then*

(i) T_0 is uniquely ergodic and has purely point spectrum;

(ii) the eigenvalues of T_0 are all nth roots of unity, where n runs through the set $\mathrm{CF}(T_0)$*;*

(iii) all eigenvalues of T_0 are simple and all eigenfunctions are continuous.

Proof. Let T_0 be the odometer on $X = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots$. Denote by μ_0 a Borel probability measure on X that is the direct product of normalized counting measures on finite sets $\mathbb{Z}_{n_1}, \mathbb{Z}_{n_2}, \dots$. For any cylindrical set C of the form $\{z\} \times \mathbb{Z}_{n_{k+1}} \times \mathbb{Z}_{n_{k+1}} \times \dots$, where $z \in \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$, we have $\mu_0(C) = (n_1 n_2 \cdots n_k)^{-1}$. For a fixed k, there are $n_1 n_2 \cdots n_k$ such sets. They partition the set X and are cyclically permuted by the odometer T_0. This implies that $\mu_0(T_0^{-1}(C)) = \mu_0(C)$ for all cylindrical sets C. Also, $\mu(C) = \mu_0(C)$ for any Borel probability measure on X invariant under T_0. Since any Borel measure on X is determined by its values on cylindrical sets, we obtain that the odometer T_0 is uniquely ergodic and μ_0 is the unique invariant measure.

Let E be the set of all nth roots of unity, where n runs through the set $\mathrm{CF}(T_0)$. Take any $\zeta \in E$. We have $\zeta^n = 1$ for some $n \in \mathrm{CF}(T_0)$. Then $\zeta = \zeta_n^k$, where $\zeta_n = e^{2\pi i/n}$ and k is a positive integer. By Lemma 2.9, the odometer T_0 admits a continuous eigenfunction f_n associated with the eigenvalue ζ_n. It is easy to observe that f_n^k is a continuous eigenfunction of T_0 associated with the eigenvalue ζ. According to Lemma 2.9, any continuous eigenfuction of T_0 is determined by its eigenvalue uniquely up to scaling.

To finish the proof of the proposition, it remains to show that the set F of continuous eigenfunctions of T_0 associated with eigenvalues from the set E is complete in the Hilbert space $L_2(X, \mu_0)$, i.e., the linear span of F is dense in $L_2(X, \mu_0)$. Take an arbitrary cylindrical set $C = \{z\} \times \mathbb{Z}_{n_{k+1}} \times \mathbb{Z}_{n_{k+1}} \times \dots$. The product $m = n_1 n_2 \cdots n_k$ belongs to $\mathrm{CF}(T_0)$ due to Lemma 2.6. Then a primitive mth root of unity $\zeta_m = e^{2\pi i/m}$ belongs to E. Let f be a continuous eigenfunction of T_0 associated with the eigenvalue ζ_m. The minimality of T_0 implies that f is nowhere zero. Replacing f by a scalar multiple, we can assume that $f(x_0) = 1$ for some $x_0 \in C$. Then $f(T_0^k x_0) = \zeta_m^k$ for $k = 1, 2, \dots$. Note that $T_0^k x_0 \in C$ if and only if k divides m. Besides, $f(T_0^k x_0) = 1$ if and only if k divides m. It follows that all values of f are mth roots of unity, moreover, $f(x) = 1$ if and only if $x \in C$. Therefore the sum $f + f^2 + \cdots + f^m$ is identically m on the set C and identically zero elsewhere. Notice that each term in this sum is a continuous eigenfunction of T_0 with eigenvalue an mth root of unity. Thus the characteristic functions of cylindrical sets are contained in the span of F. These characteristic functions form a complete set in $L_2(X, \mu)$ for any finite Borel measure μ on X. ∎

Sometimes the odometers as defined above in this section are called *generalized odometers* while the notion "odometer" refers to p-adic odometers, which are defined as follows. Let p be a prime integer and $0 < \rho < 1$. Any nonzero $r \in \mathbb{Q}$ is uniquely represented in the form $p^k \frac{m}{n}$, where k, m, n are integers, $n > 0$, and p divides neither m nor n. We let $|r|_p = \rho^k$. Also, let $|0|_p = 0$. Now $|\cdot|_p$ is a norm on the field $\mathbb{Q}$ called the *p-adic norm*. The p-adic norm induces a distance d_p on $\mathbb{Q}$, $d_p(r_1, r_2) = |r_1 - r_2|_p$ for all $r_1, r_2 \in \mathbb{Q}$. By definition, the field $\mathcal{F}_p$ of *p-adic numbers* is the completion of the field $\mathbb{Q}$ with respect to the p-adic norm. The ring $\mathcal{Z}_p$ of *p-adic integers* is the closure of the ring $\mathbb{Z}$ in $\mathcal{F}_p$. The transformation $x \mapsto x + 1$ of $\mathcal{Z}_p$ is called the *p-adic odometer*. The 2-adic odometer is also called the *binary odometer*.

Lemma 2.14 *The p-adic odometer is continuously conjugated to the odometer on $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \dots$.*

Proof. Let X denote $\mathbb{Z}_p \times \mathbb{Z}_p \times \dots$ and T denote the odometer on X. By X_0 denote the countable product $\{0, 1, \dots, p-1\} \times \{0, 1, \dots, p-1\} \times \dots$. We endow X_0 with the product topology. Then the map $f : X_0 \to X$ defined by $f(n_1, n_2, \dots) = (n_1 + p\mathbb{Z}, n_2 + p\mathbb{Z}, \dots)$ is a homeomorphism.

An arbitrary p-adic integer is uniquely expanded into a series of the form $\sum_{k=1}^{\infty} n_k p^{k-1}$, where $n_k \in \{0, 1, \dots, p-1\}$ for $k = 1, 2, \dots$. Moreover, any series of this form converges in $\mathcal{Z}_p$. It follows that the map $h : X_0 \to \mathcal{Z}_p$ defined by $h(n_1, n_2, \dots) = \sum_{k=1}^{\infty} n_k p^{k-1}$ is one-to-one and onto. It is easy to see that h is continuous as well. Since X_0 is compact, h is a homeomorphism.

It follows from the definition of the maps f and h that $hf^{-1}(Tx) = hf^{-1}(x) + 1$ for all $x \in X$. Thus T is continuously conjugated to the p-adic odometer. ∎

3 Toeplitz sequences

Let $\mathcal{A}$ be a nonempty finite set. We denote by $\mathcal{A}^{\mathbb{N}}$ the countable product $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \dots$ endowed with the product topology (here $\mathbb{N}$ refers to positive integers). Any $\omega \in \mathcal{A}^{\mathbb{N}}$ is represented as an infinite sequence $(\omega_1, \omega_2, \dots)$ of elements of $\mathcal{A}$. Denote by σ the map on $\mathcal{A}^{\mathbb{N}}$ that sends any sequence ω to the sequence obtained by deleting the first element of ω. That is, if $\omega = (\omega_1, \omega_2, \dots)$ then $\sigma\omega = (\omega'_1, \omega'_2, \dots)$, where $\omega'_k = \omega_{k+1}$ for $k = 1, 2, \dots$. The map σ is called the *(one-sided) shift*. It is a continuous map of the compact topological space $\mathcal{A}^{\mathbb{N}}$ onto itself. The restriction of the shift to any closed invariant subset of $\mathcal{A}^{\mathbb{N}}$ is called a *subshift*. In particular, any sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ gives rise to the subshift $\sigma|_{Z(\omega,\sigma)}$, where $Z(\omega,\sigma)$ is the closure of the orbit $\omega, \sigma\omega, \sigma^2\omega, \dots$.

A sequence $\omega = (\omega_1, \omega_2, \dots) \in \mathcal{A}^{\mathbb{N}}$ is called a *Toeplitz sequence* if for any positive integer n there exists a positive integer p such that $\omega_n = \omega_{n+kp}$ for $k = 1, 2, \dots$. If ω is a Toeplitz sequence, then all shifted sequences $\sigma\omega, \sigma^2\omega, \dots$ are also Toeplitz sequences.

Lemma 3.1 *Let $\omega = (\omega_1, \omega_2, \dots)$ be a Toeplitz sequence. Suppose n and p are positive integers such that $\omega_n = \omega_{n+kp}$ for $k = 1, 2, \dots$. If $p < n$ then $\omega_{n-p} = \omega_n$.*

Proof. Since ω is a Toeplitz sequence, there exists a positive integer q such that $\omega_{n-p} = \omega_{n-p+kq}$ for $k = 1, 2, \dots$. In particular, $\omega_{n-p} = \omega_{n-p+pq}$. On the other hand, $\omega_{n-p+pq} = \omega_{n+(q-1)p} = \omega_n$. ∎

Lemma 3.2 *For any Toeplitz sequence ω the subshift $\sigma|_{Z(\omega,\sigma)}$ is minimal.*

Proof. For any integer $n \geq 1$ there exists an integer $p_n \geq 1$ such that $\omega_n = \omega_{n+kp_n}$ for $k = 1, 2, \dots$. Take an arbitrary integer $N \geq 1$ and let $p = p_1 p_2 \dots p_N$. Then $\omega_n = \omega_{n+kp}$ for $1 \leq n \leq N$ and $k \geq 1$. In particular, for any $k \geq 1$ the first N elements of the sequence $\sigma^{kp}\omega$ are the same as the first N elements of ω.

Given $\omega' \in Z(\omega,\sigma)$, there are nonnegative integers $n_1, n_2, \dots$ such that $\sigma^{n_k}\omega \to \omega'$ as $k \to \infty$. It is no loss to assume that the numbers $n_1, n_2, \dots$ have the same remainder r under division by p. Then each $\sigma^{n_k+p-r}\omega$ has the same first N elements as ω. Since $\sigma^{n_k+p-r}\omega \to \sigma^{p-r}\omega'$ as $k \to \infty$, the sequence $\sigma^{p-r}\omega'$ also has the same first N elements as ω. Since N can be chosen arbitrarily large, it follows that ω is in the closure of the orbit $\omega', \sigma\omega', \sigma^2\omega', \dots$. Hence the orbit $\omega', \sigma\omega', \sigma^2\omega', \dots$ is dense in $Z(\omega,\sigma)$. ∎

Lemma 3.3 *Suppose ω is a Toeplitz sequence. Then for any integer $p > 0$ there exists an integer $K > 0$ with the following property. If for some $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega,\sigma)$ and integer $n > 0$ we have that $\omega'_n = \omega'_{n+kp}$ for $1 \leq k \leq K$, then $\omega'_n = \omega'_{n+kp}$ for all $k \geq 1$.*

Proof. For any $K \geq 1$ let $S(p,K)$ denote the set of positive integers n such that $\omega_n \neq \omega_{n+kp}$ for some $1 \leq k \leq K$. By $P(p)$ denote the set of positive integers n such that $\omega_n = \omega_{n+kp}$ for all $k \geq 1$. Take any $q \in \{1, 2, \dots, p\}$. If $q \in P(p)$ then the numbers $q+p, q+2p, \dots$ are in $P(p)$ as well. Now suppose that $q \notin P(p)$. Then $\omega_{q+kp} \neq \omega_q$ for some $k \geq 1$. Since ω is a Toeplitz sequence, there exist $p_1, p_2 > 0$ such that $\omega_{q+mp_1} = \omega_q$ and $\omega_{q+kp+mp_2} = \omega_{q+kp}$ for all $m \geq 1$. In particular, $\omega_{q+lp_1p_2p} \neq \omega_{q+kp+lp_1p_2p}$ for $l = 0, 1, 2, \dots$. It follows that each of the numbers $q, q+p, q+2p, \dots$ belongs to the set $S(p, k+p_1p_2)$.

Let $q \in \{1, 2, \dots, p\}$. By the above the numbers $q, q+p, q+2p, \dots$ either all belong to $P(p)$, or else they all belong to $S(p, K)$ for some $K \geq 1$. Hence there exists an integer $K_0 \geq 1$ such that $\mathbb{N} = P(p) \cup S(p, K_0)$.

Let $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega, \sigma)$. Suppose that for some integer $n \geq 1$ we have that $\omega'_n = \omega'_{n+kp}$ for $1 \leq k \leq K_0$. Since $\omega' \in Z(\omega, \sigma)$, there are nonnegative integers $n_1, n_2, \dots$ such that $\sigma^{n_m}\omega \to \omega'$ as $m \to \infty$. If m is large enough, then the first $n + K_0 p$ elements of the sequence $\sigma^{n_m}\omega$ are the same as the first $n + K_0 p$ elements of ω'. In particular, $\omega_{n_m+n} = \omega_{n_m+n+kp}$ for $1 \leq k \leq K_0$. This means that $n_m + n \notin S(p, K_0)$. As $\mathbb{N} = P(p) \cup S(p, K_0)$, we obtain that $n_m + n \in P(p)$, i.e., $\omega_{n_m+n} = \omega_{n_m+n+kp}$ for all $k \geq 1$. Since $\sigma^{n_m}\omega \to \omega'$ as $m \to \infty$, it follows that $\omega'_n = \omega'_{n+kp}$ for all $k \geq 1$. ∎

Lemma 3.4 *Suppose ω is a Toeplitz sequence. Then the Toeplitz sequences in $Z(\omega, \sigma)$ form a residual (dense G_δ) subset.*

Proof. Given any positive integers n, p, k, let $T(n, p, k)$ be the set of all sequences $\omega' = (\omega'_1, \omega'_2, \dots) \in \mathcal{A}^{\mathbb{N}}$ such that $\omega'_n = \omega'_{n+ip}$ for $i = 1, 2, \dots, k$. Clearly, $T(n, p, k)$ is an open subset of $\mathcal{A}^{\mathbb{N}}$. The set $\mathcal{T}$ of all Toeplitz sequences in $\mathcal{A}^{\mathbb{N}}$ can be represented as

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{k=1}^{\infty} T(n, p, k).$$

Suppose ω is a Toeplitz sequence. According to Lemma 3.3, for any $p \geq 1$ there exists an integer $K_p \geq 1$ such that

$$\bigcap_{k=1}^{\infty} T(n, p, k) \cap Z(\omega, \sigma) = T(n, p, K_p) \cap Z(\omega, \sigma)$$

for all $n \geq 1$. Then

$$\mathcal{T} \cap Z(\omega, \sigma) = \bigcap_{n=1}^{\infty} \Bigg(\bigcup_{p=1}^{\infty} T(n, p, K_p) \Bigg) \cap Z(\omega, \sigma).$$

Since $\bigcup_{p=1}^{\infty} T(n, p, K_p)$ is an open subset of $\mathcal{A}^{\mathbb{N}}$, it follows that $\mathcal{T} \cap Z(\omega, \sigma)$ is a G_δ subset of $Z(\omega, \sigma)$. It is dense in $Z(\omega, \sigma)$ since $\omega, \sigma\omega, \sigma^2\omega, \dots$ are Toeplitz sequences. ∎

Let $\omega = (\omega_1, \omega_2, \dots)$ be a Toeplitz sequence. We shall say that a positive integer p is a *partial period* of ω if there is $n \geq 1$ such that $\omega_n = \omega_{n+kp}$ for $k = 1, 2, \dots$. The integer p is called an *essential partial period* of ω if there exists $m \geq 1$ such that $\omega_m = \omega_{m+kp}$ for $k = 1, 2, \dots$ while for any $1 \leq p' < p$ the sequence $\omega_m, \omega_{m+p'}, \omega_{m+2p'}, \dots$ contains an element different from ω_m. We denote by $\mathrm{EP}(\omega)$ the set of all essential partial periods of ω (EP stands for "essential periods").

Recall from Section 2 that to each continuous self-mapping T of a compact topological space we associate the set $\mathrm{CF}(T)$ of positive integers n such that the cyclic permutation of order n is a factor of T. It turns out that essential partial periods of a Toeplitz sequence ω completely determine the set $\mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$.

Lemma 3.5 *For any Toeplitz sequence ω one has $\mathrm{EP}(\omega) \subset \mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$.*

Proof. Let $\omega = (\omega_1, \omega_2, \dots)$ be a Toeplitz sequence and $p \in \mathrm{EP}(\omega)$. Pick $m \geq 1$ such that $\omega_m = \omega_{m+kp}$ for $k = 1, 2, \dots$ while for any $1 \leq p' < p$ the sequence $\omega_m, \omega_{m+p'}, \omega_{m+2p'}, \dots$ contains an element different from ω_m.

Denote ω_m by a. For any $\omega' = (\omega_1, \omega_2, \dots) \in Z(\omega, \sigma)$ consider the set $R(\omega')$ of cosets $\alpha \in \mathbb{Z}/p\mathbb{Z}$ such that $\omega'_n = a$ for all $n \in \alpha$, $n > 0$. Lemma 3.1 implies that $m + p\mathbb{Z} \in R(\omega)$. Besides, $p' + R(\omega) \neq R(\omega)$ for any $1 \leq p' < p$ as otherwise $\omega_m = \omega_{m+p'} = \omega_{m+2p'} = \dots$ It follows that the sets $R(\omega), 1 + R(\omega), \dots, (p-1) + R(\omega)$ are all distinct.

By Lemma 3.1, $R(\sigma\omega') = -1 + R(\omega')$ for all $\omega' \in Z(\omega, \sigma)$. By Lemma 3.3, the set $R(\omega')$ is locally constant as a function of ω'. It follows that for any $\omega' \in Z(\omega, \sigma)$ we have $R(\omega') = -\alpha(\omega') + R(\omega)$, where $\alpha(\omega') \in \mathbb{Z}/p\mathbb{Z}$. Moreover, $\alpha(\omega')$ is uniquely determined by ω', the mapping $\omega' \mapsto \alpha(\omega')$ is continuous, and $\alpha(\sigma\omega') = \alpha(\omega') + 1$. Thus the odometer on $\mathbb{Z}/p\mathbb{Z}$ is a continuous factor of the subshift $\sigma|_{Z(\omega,\sigma)}$. That is, $p \in \mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$. ■

Proposition 3.6 *Suppose ω is a Toeplitz sequence. Then a positive integer n belongs to $\mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$ if and only if n divides the least common multiple of some $n_1, n_2 \dots, n_k \in \mathrm{EP}(\omega)$. Equivalently, $\mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$ is the smallest set of positive integers that contains $\mathrm{EP}(\omega)$ and satisfies assumptions (i), (ii), (iii) of Lemma 2.10.*

Proof. For any integer $n \geq 1$ let p_n be the smallest positive integer such that $\omega_n = \omega_{n+kp_n}$ for $k = 1, 2, \dots$. Clearly, $p_n \in \mathrm{EP}(\omega)$. Moreover, the sequence $p_1, p_2, \dots$ contains all essential partial periods of ω. For any $m \geq 1$ let $q_m = \mathrm{lcm}(p_1, p_2, \dots, p_m)$. Then $\omega_n = \omega_{n+kq_m}$ for all $n \in \{1, 2, \dots, m\}$ and $k \geq 1$. In particular, the first m elements of the sequence $\sigma^{q_m}\omega$ are the same as the first m elements of ω. It follows that $\sigma^{q_m}\omega \to \omega$ as $m \to \infty$.

Suppose $n \in \mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$, i.e., the subshift $\sigma|_{Z(\omega,\sigma)}$ is a continuous extension of a cyclic permutation $T : X \to X$ of order n. Let $f : Z(\omega, \sigma) \to X$ be a continuous map such that $f\sigma = Tf$. Since $\sigma^{q_m}\omega \to \omega$ as $m \to \infty$, we have that $f(\sigma^{q_m}\omega) \to f(\omega)$ as $m \to \infty$. But X is a finite set with the discrete topology so $f(\sigma^{q_m}\omega) = T^{q_m}f(\omega)$ actually coincides with $f(\omega)$ for large m. It follows that n divides q_m for large m.

Let F_0 denote the smallest set of positive integers that contains $\mathrm{EP}(\omega)$ and satisfies assumptions (i), (ii), (iii) of Lemma 2.10. Clearly, a positive integer belongs to F_0 if and only if it divides $\mathrm{lcm}(n_1, n_2, \dots, n_k)$ for some $n_1, n_2, \dots, n_k \in \mathrm{EP}(\omega)$.

By the above $\mathrm{CF}(\sigma|_{Z(\omega,\sigma)}) \subset F_0$. On the other hand, $\mathrm{EP}(\omega) \subset \mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$ due to Lemma 3.5. Then it follows from Lemma 2.8 that $F_0 \subset \mathrm{CF}(\sigma|_{Z(\omega,\sigma)})$. Thus $\mathrm{CF}(\sigma|_{Z(\omega,\sigma)}) = F_0$. ∎

Lemma 3.7 *Let ω be a Toeplitz sequence, $T : X \to X$ be a maximal odometer factor of $\sigma|_{Z(\omega,\sigma)}$, and $f : Z(\omega,\sigma) \to X$ be a continuous map such that $Tf = f\sigma$ on $Z(\omega,\sigma)$. Then, given $\omega' \in Z(\omega,\sigma)$, the equation $f(\eta) = f(\omega')$ has a solution η different from ω' if and only if ω' is not a Toeplitz sequence.*

Proof. We have $X = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots$ for some positive integers $n_1, n_2, \dots$. For any integer $k \geq 1$ let $m_k = n_1 n_2 \cdots n_k$. Given a sequence $x = (x_1, x_2, \dots) \in X$ and an integer $N \geq 1$, the first k elements of the sequence $T^N x$ match the first k elements of x if and only if m_k divides N. Consequently, $T^{N_i} x \to x$ as $i \to \infty$ if and only if each m_k divides all but finitely many of the numbers $N_1, N_2, \dots$.

First assume that $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega,\sigma)$ is not a Toeplitz sequence. Then there exists an index $m \geq 1$ such that for any integer $p \geq 1$ a subsequence $\omega'_m, \omega'_{m+p}, \omega'_{m+2p}, \dots$ is not constant. That is, $\omega'_m \neq \omega'_{m+N(p)p}$ for some $N(p) \geq 1$. Take any limit point ω'' of a sequence $\sigma^{N(m_1)m_1}\omega', \sigma^{N(m_2)m_2}\omega', \dots$. By the above, $f(\sigma^{N(m_k)m_k}\omega') = T^{N(m_k)m_k} f(\omega') \to f(\omega')$ as $k \to \infty$. Therefore $f(\omega'') = f(\omega')$. By construction, the sequences ω'' and ω' differ at least in the mth element.

Now consider the case when ω' is a Toeplitz sequence. Take any $\eta \in Z(\omega,\sigma)$ such that $f(\eta) = f(\omega')$. According to Lemma 3.2, the subshift $\sigma|_{Z(\omega,\sigma}$ is minimal. Hence $Z(\omega',\sigma) = Z(\omega,\sigma)$. Then for some integers $0 \leq N_1 \leq N_2 \leq \dots$ we have $\sigma^{N_i}\omega' \to \eta$ as $i \to \infty$, which implies that $T^{N_i} f(\omega') = f(\sigma^{N_i}\omega') \to f(\eta) = f(\omega')$ as $i \to \infty$. By the above each m_k divides all but finitely many of the numbers $N_1, N_2, \dots$. Since ω' is a Toeplitz sequence, for any index $m \geq 1$ there exists $p \in \mathrm{EP}(\omega')$ such that $\omega'_m = \omega'_{m+np}$, $n = 1, 2, \dots$. By Lemma 3.5, a cyclic permutation of order p is a continuous factor of the subshift $\sigma|_{Z(\omega,\sigma)}$. Then Lemma 2.6 implies that p divides some m_k. Hence p divides N_i for all sufficiently large i, which implies that $\omega'_{m+N_i} = \omega'_m$ for all sufficiently large i. It follows that the mth element of the sequence η equals ω'_m. As the choice of m was arbitrary, we conclude that $\eta = \omega'$. ∎

Using notation of Lemma 3.7, let Ω_0 be the set of all Toeplitz sequences in $Z(\omega,\sigma)$. Clearly, Ω_0 is invariant under the shift. According to Lemma 3.7, Ω_0 is the largest subset of $Z(\omega,\sigma)$ such that the restriction of the mapping f to Ω_0 is one-to-one. By Lemma 3.4, Ω_0 is a residual subset of $Z(\omega,\sigma)$. One might say that, generically, the dynamics of the subshift $\sigma|_{Z(\omega,\sigma)}$ is that of the odometer T. However the dynamics of this subshift as a whole can be much more complicated. In the next section we will consider a class of Toeplitz sequences ω for which the subshift $\sigma|_{Z(\omega,\sigma)}$ is as close to an odometer as it gets.

4 Substitution subshift

Given a finite alphabet $\mathcal{A}$, we denote by $\mathcal{A}^*$ the set of all finite words in this alphabet. Any $w \in \mathcal{A}^*$ is simply an arbitrary finite sequence of elements from $\mathcal{A}$. $\mathcal{A}^*$ is a monoid with respect to concatenation (the unit element is the empty word). The concatenation of a word $w \in \mathcal{A}^*$ with a sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ is also naturally defined, which gives rise to an action of the monoid $\mathcal{A}^*$ on $\mathcal{A}^{\mathbb{N}}$.

Consider an arbitrary map $\tau : \mathcal{A} \to \mathcal{A}^*$. Since $\mathcal{A}^*$ is the free monoid generated by $\mathcal{A}$, the map τ can be extended to a homomorphism of $\mathcal{A}^*$ to itself. Given a word $w \in \mathcal{A}^*$, the word $\tau(w)$ is obtained by substituting the word $\tau(a)$ for each occurence of any letter $a \in \mathcal{A}$ in w. The latter procedure applies to infinite words as well, which gives rise to a transformation of $\mathcal{A}^{\mathbb{N}}$ called a *substitution*. For convenience, we use the notation τ for both the homomorphism of $\mathcal{A}^*$ and the transformation of $\mathcal{A}^{\mathbb{N}}$. Formally, the map $\tau : \mathcal{A} \to \mathcal{A}^*$ is uniquely extended to a transformation of the set $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ such that $\tau(uw) = \tau(u)\tau(w)$ and $\tau(u\omega) = \tau(u)\tau(\omega)$ for all $u, w \in \mathcal{A}^*$ and $\omega \in \mathcal{A}^{\mathbb{N}}$.

Let τ be a substitution on $\mathcal{A}^{\mathbb{N}}$ induced by a map $\tau : \mathcal{A} \to \mathcal{A}^*$ as described above. We assume τ to be non-degenerate in that the word $\tau(b)$ is nonempty for any letter $b \in \mathcal{A}$. Suppose that for some letter $a \in \mathcal{A}$ the word $\tau(a)$ begins with a and contains more than one letter. Consider the words $a, \tau(a), \tau^2(a), \tau^3(a), \dots$. It easily follows by induction that each word in this sequence is a beginning of the next one and that each word is shorter than the next one. Therefore the finite words $a, \tau(a), \tau^2(a), \tau^3(a), \dots$ in a sense converge to an infinite sequence $\omega \in \mathcal{A}^{\mathbb{N}}$. Namely, ω is the unique infinite sequence such that each $\tau^k(a)$ is a beginning of ω. By construction, the sequence ω is a fixed point of the substitution τ. The associated subshift $\sigma|_{Z(\omega,\sigma)}$ is called a *substitution subshift*.

In this paper, we are mostly interested in a substitution τ_G over the alphabet $\mathcal{A} = \{a, b, c, d\}$ that arises in the study of the Grigorchuk group. τ_G is defined by relations

$$\tau_G(a) = aca, \quad \tau_G(b) = d, \quad \tau_G(c) = b, \quad \tau_G(d) = c.$$

This substitution has a unique invariant sequence $\omega^{(G)}$ that is the limit of finite words $\tau_G^k(a)$, $k = 1, 2, \dots$. The following lemma provides a complete description of the sequence $\omega^{(G)} = \big(\omega_1^{(G)}, \omega_2^{(G)}, \dots\big)$.

Lemma 4.1 *Let $m = 2^k(2n+1)$, where $k, n \geq 0$ are integers. If $k = 0$ then $\omega_m^{(G)} = a$. In the case $k > 0$, we have $\omega_m^{(G)} = d$, c, or b if the remainder of k under division by 3 is 0, 1, or 2, respectively.*

Proof. The first two letters of the infinite word $\omega^{(G)}$ are ac. For any letter $l \in \{b, c, d\}$ we have $\tau_G(al) = acal'$, where $l' = \tau_G(l)$ is another letter from $\{b, c, d\}$. Using inductive argument, we derive from these simple observations that $\omega_m^{(G)} = a$ if and only if m is odd. Furthermore, $\omega_{4n-2}^{(G)} = c$ and $\omega_{4n}^{(G)} = \tau_G\big(\omega_{2n}^{(G)}\big)$ for $n = 1, 2, \dots$.

Let $m = 2^k(2n+1)$, where $k, n \geq 0$ are integers. If $k = 0$ then $\omega_m^{(G)} = a$ since m is odd. If $k = 1$ then $\omega_m^{(G)} = c$. If $k > 1$ then $\omega_m^{(G)} = \tau_G^{k-1}\big(\omega_{2(2n+1)}^{(G)}\big) = \tau_G^{k-1}(c)$. It remains to notice that $\tau_G^{k-1}(c) = d$, c, or b if the remainder of k under division by 3 is 0, 1, or 2, respectively. ∎

Lemma 4.2 *$\omega^{(G)}$ is a Toeplitz sequence. The set $\mathrm{EP}\big(\omega^{(G)}\big)$ of its essential partial periods consists of all powers of 2.*

Since $\omega^{(G)}$ is a Toeplitz sequence, Lemma 3.3 applies to it. In fact, in this particular case a much stronger statement holds.

Lemma 4.3 *Let $\omega = (\omega_1, \omega_2, \dots) \in Z\big(\omega^{(G)}, \sigma\big)$. Given positive integers p and m, if $\omega_m = \omega_{m+p} = \omega_{m+2p} = \omega_{m+3p}$, then $\omega_{m+np} = \omega_m$ for all $n \geq 1$.*

Proof of Lemmas 4.2 and 4.3. For any integer $m > 0$ let $d(m)$ denote the largest nonnegative integer such that $2^{d(m)}$ divides m. Clearly, $d(m_1 m_2) = d(m_1) + d(m_2)$ and $d(m_1 + m_2) \geq \min(d(m_1), d(m_2))$. Moreover, if $d(m_1) \neq d(m_2)$ then $d(m_1 + m_2) = \min(d(m_1), d(m_2))$.

According to Lemma 4.1, any element $\omega_m^{(G)}$ of the sequence $\omega^{(G)}$ is uniquely determined by $d(m)$. In particular, $\omega_m^{(G)} \neq \omega_n^{(G)}$ whenever $|d(m) - d(n)| = 1$.

Take any integers $p, m \geq 1$. If $d(p) > d(m)$ then $d(m + np) = d(m)$ for all $n \geq 1$, which implies that $\omega_{m+np}^{(G)} = \omega_m^{(G)}$ for all $n \geq 1$.

In the case $d(p) = d(m)$, we have $d(m+p) > d(m)$. If, in addition, $d(m+p) > d(m)+1 = d(2p)$ then $d(m+3p) = d(2p) = d(m)+1$. Therefore $d(m+p) - d(p) = 1$ or $d(m+3p) - d(p) = 1$.

In the case $d(p) < d(m)$, we have $d(m+p) = d(p)$. If, in addition, $d(m) > d(p) + 1 = d(2p)$ then $d(m+2p) = d(2p) = d(p) + 1 = d(m+p) + 1$. Therefore $d(m) - d(m+p) = 1$ or $d(m+2p) - d(m+p) = 1$.

By the above the equalities $\omega_m^{(G)} = \omega_{m+p}^{(G)} = \omega_{m+2p}^{(G)} = \omega_{m+3p}^{(G)}$ cannot hold simultaneously if $d(p) \leq d(m)$.

Denote by Ω the set of all sequences $\omega = (\omega_1, \omega_2, \dots) \in \mathcal{A}^{\mathbb{N}}$ such that, given arbitrary integers $p, m > 0$, we have $\omega_{m+np} = \omega_m$ for all $n \geq 1$ whenever this holds for $1 \leq n \leq 3$. It is easy to observe that the set Ω is shift-invariant and closed. We have just shown that $\omega^{(G)} \in \Omega$. It follows that $Z\big(\omega^{(G)}, \sigma\big) \subset \Omega$, which is exactly what Lemma 4.3 states.

We proceed to the proof of Lemma 4.2. Given an integer $m \geq 1$, let $p = 2^{d(m)+1}$. Then $d(p) = d(m) + 1 > d(m)$. This implies $\omega_{m+np}^{(G)} = \omega_m^{(G)}$ for all $n \geq 1$. On the other hand, take any $1 \leq p' < p$. Then $d(p') < d(p)$ or, equivalently, $d(p') \leq d(m)$. By the above $\omega_{m+np'}^{(G)} \neq \omega_m^{(G)}$ for some $1 \leq n \leq 3$. Since m can be chosen arbitrarily, we conclude that $\omega^{(G)}$ is a Toeplitz sequence and $\mathrm{EP}\big(\omega^{(G)}\big) = \{2^{d(m)+1} \mid m \geq 1\} = \{2^k \mid k \geq 1\}$. ∎

The remainder of this section is devoted to the study of the subshift $\sigma|_{Z(\omega,\sigma)}$ associated to a Toeplitz sequence ω with $\mathrm{EP}(\omega) = \{2^k \mid k \geq 1\}$. We shall see that the latter condition completely determines the dynamics of the subshift. Note that, in general, $\sigma|_{Z(\omega,\sigma)}$ is not a substitution subshift.

Proposition 4.4 *Let ω be a Toeplitz sequence such that $\mathrm{EP}(\omega) = \{2^k \mid k \geq 1\}$. Then*

- *the preimage under the subshift $\sigma|_{Z(\omega,\sigma)}$ of any sequence $\omega' \in Z(\omega,\sigma)$ is nonempty;*
- *there exists a unique sequence $\omega^* \in Z(\omega,\sigma)$ for which this preimage contains more than one sequence;*
- *a sequence $\omega' \in Z(\omega,\sigma)$ is not a Toeplitz sequence if and only if $\sigma^n\omega' = \omega^*$ for some $n \geq 1$.*

In the case $\omega = \omega^{(G)}$, we have $\omega^* = \omega^{(G)}$. The preimage of $\omega^{(G)}$ under the subshift consists of three sequences $b\omega^{(G)}$, $c\omega^{(G)}$, and $d\omega^{(G)}$.

Proof. Let $\omega = (\omega_1, \omega_2, \dots)$. Take an arbitrary integer $k \geq 1$. Since $2^k \in \mathrm{EP}(\omega)$, there exist a letter $l_k \in \mathcal{A}$ and an index $m_k \geq 1$ such that $\omega_{m_k+2^k n} = l_k$, $n = 0, 1, 2, \dots$, but for any $1 \leq p < 2^k$ the sequence $\omega_{m_k}, \omega_{m_k+p}, \omega_{m_k+2p}, \dots$ contains an element different from l_k. Lemma 3.1 implies that $\omega_n = l_k$ for every positive $n \equiv m_k \bmod 2^k$. It follows that $m_K \not\equiv m_k \bmod 2^k$ if $K > k$. As a consequence, the congruence classes $m_1 + 2\mathbb{Z}, m_2 + 2^2\mathbb{Z}, \dots$ are pairwise disjoint.

Let $U_0 = \mathbb{Z}$ and $U_k = U_{k-1} \setminus (m_k + 2^k\mathbb{Z})$ for $k \geq 1$. Since $m_1 + 2\mathbb{Z}, m_2 + 2^2\mathbb{Z}, \dots$ are pairwise disjoint sets, it follows that each U_k is a congruence class modulo 2^k. That is, $U_k = M_k + 2^k\mathbb{Z}$ for a unique $1 \leq M_k \leq 2^k$. By construction, $M_k \equiv M_{k+1} \bmod 2^k$. Hence $M_{k+1} = M_k$ or $M_{k+1} = M_k + 2^k$. In particular, the sequence $M_1, M_2, \dots$ is nondecreasing. Note that $\omega_{m+2^k} = \omega_m$ for any positive integer $m \not\equiv M_k \bmod 2^k$. This implies that the sequence $\omega_{M_k}, \omega_{M_k+2^k}, \dots, \omega_{M_k+2^k n}, \dots$ contains an element different from ω_{M_k} as otherwise ω will be periodic (with period 2^k), which is impossible since $\mathrm{EP}(\omega)$ is an infinite set. On the other hand, since all essential partial periods of ω are powers of 2, we do have $\omega_{M_k+2^K n} = \omega_{M_k}$ for some $K > k$ and $n = 1, 2, \dots$. Then $M_k < M_K$. Hence the sequence $M_1, M_2, \dots$ tends to infinity. As a consequence, the intersection of sets $U_0, U_1, U_2, \dots$ contains no positive integer (and at most one nonpositive integer).

The sequences $l_1, l_2, \dots$ and $M_1, M_2, \dots$ can be used to reconstruct the sequence ω. Namely, $\omega_m = l_k$ whenever $m \in U_{k-1} \setminus U_k$, that is, $m \not\equiv M_k \bmod 2^k$ while $m \equiv M_{k-1} \bmod 2^{k-1}$. Observe that the sequence of letters $l_1, l_2, \dots$ is not eventually constant as otherwise ω would be periodic.

Now consider an arbitrary sequence $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega,\sigma)$. We have $\sigma^{n_i}\omega \to \omega'$ as $i \to \infty$ for some $0 \leq n_1 \leq n_2 \leq \dots$. Since $2^k \in \mathrm{EP}(\omega)$ for any $k \geq 1$, it follows from Lemma 3.5 that a cyclic permutation of order 2^k

is a continuous factor of the subshift $\sigma|_{Z(\omega,\sigma)}$. As a consequence, the sequences $\sigma^{n_1}\omega, \sigma^{n_2}\omega, \dots$ may converge in $Z(\omega,\sigma)$ only if the numbers $n_1, n_2, \dots$ eventually have the same remainder r_k under division by 2^k. Furthermore, the remainder r_k depends only on the limit sequence ω' and not on the choice of n_i. Therefore we have a unique, well defined integer $1 \le M_k(\omega') \le 2^k$ such that $M_k(\omega') + r_k \equiv M_k \bmod 2^k$. Observe that $M_k(\omega) = M_k$ and $M_k(\sigma\omega') + 1 \equiv M_k(\omega') \bmod 2^k$. By construction, $r_{k+1} \equiv r_k \bmod 2^k$, which implies that $M_{k+1}(\omega') \equiv M_k(\omega') \bmod 2^k$. In other words, the congruence classes $M_k(\omega') + 2^k\mathbb{Z}$, $k \ge 1$ are nested just as in the case $\omega' = \omega$. There is more similarity. For example, we know that a sequence $\omega_m, \omega_{m+2^k}, \dots, \omega_{m+2^k n}, \dots$ is constant whenever $m \not\equiv M_k \bmod 2^k$, while it is not constant for $m = M_k$. It follows that a sequence $\omega'_m, \omega'_{m+2^k}, \dots, \omega'_{m+2^k n}, \dots$ is constant whenever $m \not\equiv M_k(\omega') \bmod 2^k$. On the other hand, this sequence is not constant for $m = M_k(\omega')$ as otherwise ω' will be periodic (with period 2^k), which is impossible since the orbit $\omega', \sigma\omega', \sigma^2\omega', \dots$ is dense in the infinite set $Z(\omega,\sigma)$ due to Lemma 3.2. Actually, the only difference of the general case from the case $\omega' = \omega$ is that the sequence $M_1(\omega'), M_2(\omega'), \dots$ may not tend to infinity.

The sequences $l_1, l_2, \dots$ and $M_1(\omega'), M_2(\omega'), \dots$ can be used to reconstruct the sequence ω'. We know that $\omega_m = l_k$ whenever $m \not\equiv M_k \bmod 2^k$ and $m \equiv M_{k-1} \bmod 2^{k-1}$. It follows that $\omega'_m = l_k$ whenever $m \not\equiv M_k(\omega') \bmod 2^k$ and $m \equiv M_{k-1}(\omega') \bmod 2^{k-1}$. If $M_k(\omega') \to \infty$ as $k \to \infty$, then this is enough to reconstruct ω'. Also, this implies that ω' is a Toeplitz sequence. Otherwise, when the sequence $M_1(\omega'), M_2(\omega'), \dots$ is eventually constant, we still need to determine an element ω_{M_∞}, where M_∞ is the limit of $M_k(\omega')$ as $k \to \infty$. By the way, M_∞ completely determines the sequence $M_1(\omega'), M_2(\omega'), \dots$ as $M_k(\omega') \equiv M_\infty \bmod 2^k$, $k \ge 1$. Let $l_\infty(\omega') = \omega_{M_\infty}$. Along with the letter $l_\infty(\omega')$, the sequences $l_1, l_2, \dots$ and $M_1(\omega'), M_2(\omega'), \dots$ determine ω' uniquely. Notice that the letter $l_\infty(\omega')$ must occur infinitely often in the sequence $l_1, l_2, \dots$. Indeed, assume that some $a \in \mathcal{A}$ never occurs in a subsequence $l_k, l_{k+1}, \dots$. Then $\omega_m = a$ implies $\omega_{m+2^{k-1}} = a$ for any $m \ge 1$. Consequently, $\omega'_m = a$ implies $\omega'_{m+2^{k-1}} = a$ for any $m \ge 1$. It follows that $\omega'_{M_K(\omega')} \ne a$ for all $K \ge k-1$.

Next consider an arbitrary sequence of integers $M'_1, M'_2, \dots$ such that $1 \le M'_k \le 2^k$ and $M'_{k+1} = M'_k \bmod 2^k$ for all $k \ge 1$. We associate to it another sequence of positive integers $R_1, R_2, \dots$ defined by $R_k = M_k - M'_k + 2^k$. For any $k \ge 1$ the numbers $R_k, R_{k+1}, \dots$ have the same remainder r_k under division by 2^k. Clearly, $M'_k + r_k \equiv M_k \bmod 2^k$. It follows from the above that any limit point ω' of the sequence $\sigma^{R_1}\omega, \sigma^{R_2}\omega, \dots$ satisfies $M_k(\omega') = M'_k$ for all $k \ge 1$. In particular, there exists a sequence $\omega^* \in Z(\omega,\sigma)$ such that $M_k(\omega^*) = 2^k$, $k = 1, 2, \dots$. Such a sequence is unique since $2^k \to \infty$ as $k \to \infty$.

Further, assume that the sequence $M'_1, M'_2, \dots$ from the previous paragraph is eventually constant and denote by M_∞ its limit. By the above there is at least one $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega,\sigma)$ such that $M_k(\omega') = M'_k$ for all $k \ge 1$. Pick any letter $l \in \mathcal{A}$ that occurs infinitely often in the sequence $l_1, l_2, \dots$. Let ω'' be the sequence obtained from ω' by substituting l for a single element ω'_{M_∞}. We shall show that $\omega'' \in Z(\omega, \sigma)$. Given an integer $k \geq 1$, a sequence $\omega'_m, \omega'_{m+2^k}, \dots, \omega'_{m+2^k n}, \dots$ is constant whenever $m \not\equiv M_k(\omega') \equiv M_\infty \bmod 2^k$. On the other hand, in the case $m = M_\infty$ this sequence is not constant as it contains all letters that occur in the sequence $l_{k+1}, l_{k+2}, \dots$. In particular, $\omega'_{M_\infty+2^k n} = l$ for some $n \geq 0$. Then the first 2^k elements of the shifted sequence $\sigma^{2^k n}\omega'$ coincide with the first 2^k elements of ω''. Since k can be chosen arbitrarily large, it follows that $\omega'' \in Z(\omega, \sigma)$. Note that $M_k(\omega'') = M_k(\omega') = M'_k$ for all $k \geq 1$ and that $l_\infty(\omega'') = l$.

We already know that a sequence $\omega' \in Z(\omega, \sigma)$ is a Toeplitz sequence if $M_k(\omega') \to \infty$ as $k \to \infty$. Let us show that ω' is not a Toeplitz sequence if the sequence $M_1(\omega'), M_2(\omega'), \dots$ is eventually constant. As follows from the above, in this case there exists a sequence $\omega'' \in Z(\omega, \sigma)$ that differs from ω' in a single element. Then $\sigma^n\omega' = \sigma^n\omega''$ for some $n \geq 1$. Now we are going to apply Lemma 3.7. Let $T : X \to X$ be a maximal odometer factor of the subshift $\sigma|_{Z(\omega,\sigma)}$ and $f : Z(\omega, \sigma) \to X$ be a continuous map such that $Tf = f\sigma$ on $Z(\omega, \sigma)$. Since $\sigma^n\omega' = \sigma^n\omega''$, we obtain $T^n(f(\omega')) = T^n(f(\omega''))$. But every odometer is a one-to-one mapping, which implies that $f(\omega') = f(\omega'')$. By Lemma 3.7, neither ω' nor ω'' is a Toeplitz sequence.

Recall that $M_k(\sigma\omega') + 1 \equiv M_k(\omega') \bmod 2^k$ for all $k \geq 1$ and $\omega' \in Z(\omega, \sigma)$. Consequently, for any $n \geq 1$ we have $M_k(\sigma^n\omega') + n \equiv M_k(\omega') \bmod 2^k$. It follows that the sequence $M_1(\omega'), M_2(\omega'), \dots$ has a finite limit n if and only if $M_k(\sigma^n\omega') = 2^k$ for all $k \geq 1$. An equivalent condition is that $\sigma^n\omega' = \omega^*$. Thus a sequence $\omega' \in Z(\omega, \sigma)$ is not a Toeplitz sequence if and only if $\sigma^n\omega' = \omega^*$ for some $n \geq 1$.

By the above two sequences $\omega', \omega'' \in Z(\omega, \sigma)$ coincide if and only if $M_k(\omega') = M_k(\omega'')$ for $k = 1, 2, \dots$ and either the sequence $M_1(\omega'), M_2(\omega'), \dots$ tends to infinity, or else it is eventually constant and $l_\infty(\omega') = l_\infty(\omega'')$. Take any $\omega' \in Z(\omega, \sigma)$ and consider a unique sequence of integers $M'_1, M'_2, \dots$ such that $1 \leq M'_k \leq 2^k$ and $M_k(\omega') + 1 \equiv M'_k \bmod 2^k$ for all $k \geq 1$. The congruency classes $M'_k + 2^k\mathbb{Z}$, $k \geq 1$ are nested since the congruency classes $M_k(\omega') + 2^k\mathbb{Z}$, $k \geq 1$ are nested. As shown above, this implies the existence of an $\omega'' \in Z(\omega, \sigma)$ such that $M_k(\omega'') = M'_k$, $k \geq 1$. Let S be the set of all such ω''. The set S contains a single point if $M'_k \to \infty$ as $k \to \infty$. Otherwise S contains several points; they are distinguished by $l_\infty(\omega'')$, which can be any letter that occurs infinitely often in the sequence $l_1, l_2, \dots$. Clearly, a sequence $\omega'' \in Z(\omega, \sigma)$ belongs to S if and only if $M_k(\sigma\omega'') = M_k(\omega')$ for all $k \geq 1$. If $M_k(\omega') \to \infty$ as $k \to \infty$, then S is exactly the preimage of ω' under the subshift $\sigma|_{Z(\omega,\sigma)}$. If, in addition, $\omega' \neq \omega^*$, then $M'_1, M'_2, \dots$ tend to infinity as well so that the set S consists of a single point. In the case $\omega' = \omega^*$, all M'_k are equal to 1 so that S contains more than one point. Finally, consider the case when the sequence $M_1(\omega'), M_2(\omega'), \dots$ is eventually constant. Then the sequence $M'_1, M'_2, \dots$ is also eventually constant. For any $\omega'' \in S$ we have $l_\infty(\sigma\omega'') = l_\infty(\omega'')$. Hence the preimage of ω' under the subshift consists of a unique $\omega'' \in S$ such that $l_\infty(\omega'') = l_\infty(\omega')$.

The general part of the proposition is proved. It remains to consider a particular case $\omega = \omega^{(G)}$. According to Lemma 4.1, in this case the sequence $l_1, l_2, \dots$ is eventually periodic with period 3: $a, c, b, d, c, b, d, \dots$. Moreover, $M_k = 2^k$ for any $k \geq 1$. This means that $\omega^{(G)} = \omega^*$. By the above the preimage of $\omega^{(G)}$ under the subshift $\sigma|_{Z(\omega^{(G)},\sigma)}$ consists of all sequences of the form $l\omega^{(G)}$, where $l \in \mathcal{A}$ is a letter that occurs infinitely often in the sequence $l_1, l_2, \dots$. We have three such letters: b, c, and d. ∎

Theorem 4.5 *Let ω be a Toeplitz sequence such that $\mathrm{EP}(\omega) = \{2^k \mid k \geq 1\}$. Then the maximal odometer factor of the subshift $\sigma|_{Z(\omega,\sigma)}$ is the binary odometer.*

Moreover, the subshift $\sigma|_{Z(\omega,\sigma)}$ is, up to a countable set, continuously conjugated to the binary odometer. To be precise, there exist a countable set $\Omega_1 \subset Z(\omega,\sigma)$ and a continuous mapping f of $Z(\omega,\sigma)$ onto the ring $\mathcal{Z}_2$ of dyadic integers such that $f(\sigma\omega') = f(\omega') + 1$ for all $\omega' \in Z(\omega,\sigma)$, the complement $Z(\omega,\sigma) \setminus \Omega_1$ is shift invariant and f is one-to-one when restricted to $Z(\omega,\sigma) \setminus \Omega_1$.

Proof. Proposition 3.6 implies that $\mathrm{CF}\big(\sigma|_{Z(\omega,\sigma)}\big) = \{2^k \mid k \geq 0\}$. That is, a nontrivial cyclic permutation is a continuous factor of the subshift $\sigma|_{Z(\omega,\sigma)}$ if and only if its order is a power of 2. Let T_0 denote the odometer on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots$. By Lemma 2.6, $\mathrm{CF}(T_0) = \{2^k \mid k \geq 0\} = \mathrm{CF}\big(\sigma|_{Z(\omega,\sigma)}\big)$. Then it follows from Proposition 2.12 that T_0 is a maximal odometer factor of $\sigma|_{Z(\omega,\sigma)}$. Finally, T_0 is continuously conjugated to the binary odometer due to Lemma 2.14.

Thus there exists a continuous onto mapping $f : Z(\omega,\sigma) \to \mathcal{Z}_2$ such that $f(\sigma\omega') = f(\omega') + 1$ for all $\omega' \in Z(\omega,\sigma)$. Denote by Ω_0 the set of all Toeplitz sequences in $Z(\omega,\sigma)$. Obviously, the set Ω_0 is shift invariant. According to Lemma 3.7, the map f is one-to-one when restricted to Ω_0. By Proposition 4.4, there exists a sequence $\omega^* \in Z(\omega,\sigma)$ such that any $\omega' \in \Omega_1 = Z(\omega,\sigma) \setminus \Omega_0$ satisfies $\sigma^n\omega' = \omega^*$ for some $n \geq 1$. For any fixed n there are only finitely many sequences ω' satisfying this relation. Therefore Ω_1 is a countable set. ∎

Theorem 4.5 applies to the sequence $\omega^{(G)}$. Hence there exists a continuous map $f_G : Z(\omega^{(G)},\sigma) \to \mathcal{Z}_2$ such that $f_G(\sigma\omega') = f_G(\omega') + 1$ for all $\omega' \in Z(\omega^{(G)},\sigma)$. According to Lemma 2.3, we can choose this map so that $f_G(\omega^{(G)}) = 0$; then it is uniquely determined. Recall that any dyadic integer $z \in \mathcal{Z}_2$ is uniquely expanded into a series $\sum_{i=1}^{\infty} n_i 2^{i-1}$, where $n_i \in \{0,1\}$. Therefore the map f_G can be regarded as a symbolic map that takes an infinite word over the alphabet $\mathcal{A}$ and assigns to it an infinite word over the alphabet $\{0,1\}$. The proof of Proposition 4.4 suggests an algorithm for effective computation of f_G. Take an arbitrary sequence $\omega' = (\omega'_1, \omega'_2, \dots) \in Z(\omega^{(G)},\sigma)$. For any integer $k \geq 1$ there exists a unique integer $1 \leq M_k \leq 2^k$ such that the subsequence $\omega'_{M_k}, \omega'_{M_k+2^k}, \dots, \omega'_{M_k+2^k n}, \dots$ is not constant. Note that $0 \leq 2^k - M_k < 2^k$. The dyadic expansion of the number $2^k - M_k$ gives us the first k coefficients in the dyadic expansion of $f_G(\omega')$. That is, if $f_G(\omega') = \sum_{i=1}^{\infty} n_i 2^{i-1}$, where $n_i \in \{0,1\}$, then $2^k - M_k = \sum_{i=1}^{k} n_i 2^{i-1}$. By

Lemma 4.3, $\omega'_{M_k+2^k n} \neq \omega'_{M_k}$ already for some $1 \leq n \leq 3$. It follows that the first k coefficients in the dyadic expansion of $f_G(\omega')$ depend only on the first 2^{k+2} elements of the sequence ω'.

Theorem 4.6 *Let ω be a Toeplitz sequence such that* $\mathrm{EP}(\omega) = \{2^k \mid k \geq 1\}$. *Then*

- *there exists a unique Borel probability measure μ on $Z(\omega, \sigma)$ invariant under the subshift $\tilde{\sigma} = \sigma|_{Z(\omega,\sigma)}$.*
- *The subshift $\tilde{\sigma}$ is ergodic with respect to the measure μ. Moreover, every orbit of $\tilde{\sigma}$ is uniformly distributed with respect to μ.*
- *$\tilde{\sigma}$ has purely point spectrum, the eigenvalues being all roots of unity of order $1, 2, \ldots, 2^k, \ldots$. Each eigenvalue is simple.*
- *All eigenfunctions of $\tilde{\sigma}$ are continuous.*

Proof. Let Ω_1 be the smallest subset of $Z(\omega, \sigma)$ that contains all non-Toeplitz sequences in $Z(\omega, \sigma)$ and is both forward and backward invariant under the subshift $\tilde{\sigma}$. Proposition 4.4 implies that the set Ω_1 is countable. Denote by Ω_0 the complement of Ω_1 in $Z(\omega, \sigma)$.

By Theorem 4.5, the binary odometer is a maximal odometer factor of $\tilde{\sigma}$. Let $f : Z(\omega, \sigma) \to \mathcal{Z}_2$ be a continuous function such that $f(\sigma\omega') = f(\omega') + 1$ for all $\omega' \in Z(\omega, \sigma)$. Lemma 3.7 implies that f is one-to-one when restricted to Ω_0. Let $X_0 = f(\Omega_0)$ and denote by h the inverse of the restriction of f to Ω_0. By construction, the countable set $X_1 = f(\Omega_1)$ is the complement of X_0 in $\mathcal{Z}_2$. It follows that the mapping $h : X_0 \to \Omega_0$ is continuous. By Proposition 2.13, there exists a unique Borel probability measure μ_0 on $\mathcal{Z}_2$ that is invariant under the binary odometer. For any Borel set $B \subset Z(\omega, \sigma)$ let $\mu(B) = \mu_0(h^{-1}(B \cap \Omega_0))$. Then μ is a Borel measure on $Z(\omega, \sigma)$ invariant under the subshift $\tilde{\sigma}$. Note that $\mu_0(X_1) = 0$ since X_1 is a countable set and the binary odometer has no finite orbits. Therefore μ is a probability measure.

Suppose that μ' is a Borel probability measure on $Z(\omega, \sigma)$ invariant under the subshift $\tilde{\sigma}$. For any Borel set $b \subset \mathcal{Z}_2$ let $\mu'_0(b) = \mu'(f^{-1}(b))$. It is easy to check that μ'_0 is a Borel probability measure on $\mathcal{Z}_2$ invariant under the odometer. Hence $\mu'_0 = \mu_0$. It follows that μ' coincides with μ on the set Ω_0. Since Ω_1 is a countable set and the subshift $\tilde{\sigma}$ has no finite orbits, we obtain $\mu'(\Omega_1) = \mu(\Omega_1) = 0$ so that $\mu' = \mu$. Thus μ is the only shift-invariant Borel probability measure on $Z(\omega, \sigma)$. It follows that the subshift $\tilde{\sigma}$ is ergodic with respect to μ, moreover, each orbit of $\tilde{\sigma}$ is uniformly distributed in $Z(\omega, \sigma)$.

Since $\mu(\Omega_1) = \mu_0(X_1) = 0$, the subshift $\tilde{\sigma}$ and the binary odometer T_0 are isomorphic as measure-preserving transformations. Then Proposition 2.13 implies that $\tilde{\sigma}$ has pure point spectrum, the eigenvalues of $\tilde{\sigma}$ are all nth roots of unity, where n runs through $\mathrm{CF}(T_0)$, and all eigenvalues are simple. The set $\mathrm{CF}(T_0)$ coincides with $\mathrm{CF}(\tilde{\sigma})$ due to Proposition 2.12. Further, $\mathrm{CF}(\tilde{\sigma}) = \{2^k \mid k \geq 0\}$ due to Proposition 3.6.

To prove that all eigenfunctions of $\tilde{\sigma}$ are continuous, it is enough to show that for any root of unity ζ of order 2^k there exists an associated continuous eigenfuction of $\tilde{\sigma}$. Let $\zeta_0 = \exp(2\pi i/2^k)$. Then $\zeta = \zeta_0^m$ for some $m \geq 1$. By Lemma 2.9, there is a continuous eigenfunction ϕ of $\tilde{\sigma}$ associated with the eigenvalue ζ_0. Then ϕ^m is also a continuous eigenfunction and its eigenvalue is $\zeta_0^m = \zeta$. ∎

Proof of Theorems 1.1 and 1.2. According to Lemma 4.2, the sequence $\omega^{(G)}$ is a Toeplitz sequence with $\mathrm{EP}(\omega^{(G)}) = \{2^k \mid k \geq 1\}$. Therefore Theorem 1.1 is a particular case of Theorem 4.5 while Theorem 1.2 is a particular case of Theorem 4.6. ∎

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