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Planar structures and billiards in rational polygons: the Veech alternative

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§1. Introduction

In various problems in geometry and dynamics there naturally appears a certain geometric object on a two-dimensional surface which, depending on the context, is called a planar structure, a quadratic differential, or a measured foliation.  

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Measured foliations appear in the study of diffeomorphisms and foliations on surfaces, quadratic differentials are one of the objects of Teichmüller theory, and planar structures are closely connected with billiards in rational polygons and interval exchange transformations.

In the present article we will discuss planar structures. A planar structure on a two-dimensional compact orientable surface is a metric of zero curvature having finitely many singular points, at each of which it has a conical singularity with angle that is a multiple of $2\pi$. The progress in the study of such structures during the last 10-15 years is connected mainly with the names of W. Veech and H. Masur and has been achieved by an extensive use of methods from Teichmüller theory. Here, the planar structures themselves occur as a convenient geometric representation for quadratic differentials.

The present paper is devoted to a remarkable property of planar structures first noted by Veech [1]. We consider the simplest example of a planar structure: the torus $\mathbb{R}^2/\mathbb{Z}^2$. In this case the trajectories of the geodesic flow behave particularly simply: each of them is either periodic or uniformly distributed. It turns out that there are many examples of more complicated planar structures having this property (which will be called the Veech alternative in the sequel). More precisely, such are the planar structures having a rich group of affine symmetries. A large number of examples arise in the study of billiard flows in rational polygons, and in this case the Veech alternative provides detailed information on the dynamical properties of the corresponding flows.

The author has made an attempt to circumvent the use of methods from Teichmüller theory, making planar structures the subject of independent investigation. In this way a new proof has been obtained of Veech’s theorem [1], establishing the above-mentioned alternative. Furthermore, a new proof of a lemma of Masur [10], Theorem 1.1, has been obtained. This lemma (and the weaker assertion from [4] known before it) plays an important role in practically all investigations in this field. Finally, by developing a geometric approach to the study of planar structures it has become possible to obtain numerous results and examples related to the Veech alternative.

The structure of the paper is as follows. In §2 we give detailed preliminary information related to planar structures, billiards in rational polygons and interval exchange transformations, and also establish a link between these objects. In §3 we give a new proof of Veech’s theorem and a new proof of Masur’s lemma. We also consider the problem of the distribution of the periodic trajectories of a planar structure. In §4 we give numerous examples of Veech’s theorem, both those found by Veech himself and new ones. In §5 we consider covers of planar structures. The results obtained here substantially extend the list of examples of Veech’s theorem. Finally, in §6 we derive certain geometric properties of planar structures. A detailed study of these leads to a generalization of Veech’s theorem.

The idea of removing from the study of dynamical properties of billiards in rational polygons the non-constructivity connected with the use of Teichmüller theory is due to A. M. Stepin. The author thanks him for his universal support. The author also thanks D. V. Anosov for stimulating discussions.

The results of this paper have been announced in [18].

§2. Definition

2.1. Planar structures

Definition 2.1. Let $M$ be a closed orientable surface. A planar structure on $M$ is an atlas $\mathcal{F} = \{U_0, U_1, \ldots, U_k\}$ satisfying the following conditions:

(i) the domains $U_i$ of the atlas are disjoint.

(ii) all coordinate charts $\phi_i : U_i \to \mathbb{R}^2$ are affine.

(iii) the atlas $\mathcal{F}$ is made up of a union of open sets $U_1, U_2, \ldots, U_k$ such that each set $\{x \in U_i \mid x_i = 0\}$ contains other non-trivial open sets $U_j$. The sets $U_i \setminus \{x \in U_i \mid x_i = 0\}$ are preimages. This means that the metric is invariant under $\{x \in U_i \mid x_i = 0\}$.

A singular point of multiplicity $m$ is called a singular point of multiplicity $m$. A singular point of multiplicity $m$ is called a singular point of multiplicity $m$.

Using the charts of such an atlas, we define a Riemannian metric of zero curvature on $M$. This metric has a conical singularity at each singular point and is isometric to the metric $ds^2 = dr^2 + m \cdot (dx_1^2 + \cdots + dx_n^2)$, where $\{x_1, \ldots, x_n\}$ are local coordinates containing other non-trivial open sets $U_j$. The group of this metric is the group of isometries of this Riemannian metric called the Lebesgue measure.

The metric on $M$ induced by this flow to a level surface $\{x \mid x_i = 0\}$. Then the phase space of this flow is the set of unit vectors at an arbitrary point of $M$. If $\mu$ is an element of $\Phi$ then $\mu \times \lambda$ (with $\lambda$ the Lebesgue measure) is called the Lebesgue measure $\mu \times \lambda$.

The phase space $\Phi$ fibres over $M$. Thus, the result can be viewed as a flow on $M$; the Lebesgue measure is defined on a subset of $M$ that is invariant under this flow.

Definition 2.2. A flow on $M$ is called a Lebesgue measure if it is a flow that is invariant under this flow.
§2. Definitions and preliminary information

2.1. Planar structures.

Definition 2.1. Let \( M \) be a compact, connected, orientable surface. A planar structure on \( M \) is an atlas \( \omega \), consisting of charts of the form \( (U, f) \), where \( U \) is a domain on \( M \) and \( f \) is a homeomorphism from \( U \) to a domain in \( \mathbb{R}^2 \), such that the following conditions hold:

(i) the domains \( U \) cover the whole surface \( M \) except for finitely many points \( x_1, \ldots, x_k \), called singular;

(ii) all coordinate changing functions are shifts in \( \mathbb{R}^2 \);

(iii) the atlas \( \omega \) is maximal with respect to (i) and (ii);

(iv) for each singular point \( x_i \) there are a punctured neighbourhood \( U_i \), not containing other singular points, and a map \( f_i \) from this neighbourhood to a punctured neighbourhood \( V \) of a point in \( \mathbb{R}^2 \) that is a shift in the local coordinates from \( \omega \) and is such that each point in \( V \) has exactly \( m_i \) preimages. This number \( m_i \) is called the multiplicity of the singular point \( x_i \).

A singular point of multiplicity 1 is called removable (one can then find a planar structure \( \tilde{\omega} \supset \omega \) in which this point is non-singular).

Using the charts of the atlas \( \omega \) we can lift the Euclidean metric of \( \mathbb{R}^2 \) to a Riemannian metric of zero curvature on \( M \setminus \{x_1, \ldots, x_k\} \). At a singular point \( x_i \) this metric has a conical singularity with angle \( 2\pi m_i \), that is, a neighbourhood of this point is isometric to a neighbourhood of the origin in \( \mathbb{R}^2 \) with a metric that has the form \( ds^2 = dr^2 + (m_i r \, d\theta)^2 \) in polar coordinates \( (r, \theta) \). By (ii), the holonomy group of this metric is trivial. The area element \( \mu \) corresponding to this metric is called the Lebesgue measure on \( M \).

The metric on \( M \) induces the geodesic flow \( \{T^t\} \). We restrict our consideration of this flow to a level surface of the energy corresponding to motion with unit velocity. Then the phase space of the flow is a direct product, \( \Phi = M \times S^1 \). Here, \( S^1 \) is the circle of unit directions in \( \mathbb{R}^2 \) and can be identified with the space of unit tangent vectors at an arbitrary non-singular point. The geodesic flow is well defined on an element of \( \Phi \) if the trajectory corresponding to the element (a geodesic curve) does not pass through a singular point. Such elements form a set of full measure \( \mu \times \lambda \) (with \( \lambda \) the Lebesgue measure on \( S^1 \)). The geodesic flow preserves the measure \( \mu \times \lambda \).

The phase space \( \Phi \) fibres into invariant surfaces \( M \times \{\bar{v}\} \), which are homeomorphic to \( M \). Thus, the restriction of the geodesic flow to an invariant surface can be viewed as a flow on \( M \): the flow with unit velocity in the direction \( \bar{v} \). This flow is defined on a subset of \( M \) (depending on \( \bar{v} \)) of full Lebesgue measure and preserves this measure.

Definition 2.2. A flow on \( M \) in the direction \( \bar{v} \) is called strongly ergodic if the Lebesgue measure is the unique (up to scalar multiples) finite Borel measure on \( M \) that is invariant under the flow.

In the present case, strong ergodicity is also known as ‘unique ergodicity’.
It has been shown by Kerckhoff, Masur and Smillie [5] that for almost all $\vec{v} \in S^1$ the flow in the direction $\vec{v}$ is strongly ergodic. In particular, for the geodesic flow the decomposition into invariant surfaces is also a decomposition into ergodic components (see [11]).

We conclude this subsection by noting that in the study of quadratic differentials there appear planar structures that have a much more complicated form than the ones defined above (see [1, 5]). In fact, for a chart of the atlas $\omega$ one allows coordinate transformations of the form $\vec{v} \mapsto \pm \vec{v} + \vec{v}_0$. In this connection, the planar structures in the sense of the above definition are called orientable.

The metric on $M$ determined by a non-orientable planar structure has a non-trivial holonomy group. As a result, the geodesic flow on $M$ can, for a certain element $\vec{v}$, only be locally defined; globally it is defined as the (non-orientable) geodesic foliation. The condition of invariance of the Lebesgue measure under the flow is replaced by the condition that it be transversally invariant with respect to the foliation.

A surface with a non-orientable planar structure can be two-sheetedly covered by a surface with an orientable planar structure in a natural manner. The majority of results given below for planar structures can be transferred to the non-orientable case using this cover. Moreover, in the study of billiards in rational polygons only orientable planar structures arise. These are the subject of the next subsection.

2.2. Billiards in rational polygons. Let $Q$ be a polygon in the Euclidean plane $\mathbb{R}^2$, not necessarily convex or simply-connected. A billiard in $Q$ is a dynamical system generated by a frictionless motion of a point-ball inside $Q$ with elastic reflections in the boundary $\partial Q$. The velocity of the ball is taken to be equal to one. The motion is not restricted in time, provided that the ball does not hit a vertex of the polygon. In the opposite case the motion is defined up to the time of hitting the vertex. The phase space $\Phi(Q)$ of this dynamical system can be obtained from the direct product $Q \times S^1$ (where $S^1$ is the circle of unit directions) by identifying pairs of the form $(q, \vec{v})$ and $(q, -\vec{v})$, where $q$ is a point on a side of $Q$ and $\vec{v}, -\vec{v} \in S^1$ are vectors lying symmetrically with respect to this side. $\Phi(Q)$ inherits from $Q \times S^1$ the measure $\mu \times \lambda$ (with $\mu$ the Lebesgue measure on $Q$ and $\lambda$ the Lebesgue measure on $S^1$). The billiard flow $\{T_t\}$ is defined for all $t$ in a set of full measure $\mu \times \lambda$; it also preserves the latter measure.

Let $a$ be a side of $Q$, $r_a$ the planar symmetry with respect to this side, and $r_a$ the linear part of the operator $r_a$. Further, let $R$ be the subgroup of $O(2)$ generated by the operators $r_a$.

Definition 2.3. The polygon $Q$ is called rational if $R$ is a finite group. For simply-connected $Q$ this condition is equivalent to all angles being commensurable with $\pi$.

A construction of Zemlyakov and Katok [12] reduces a billiard in a rational polygon to the geodesic flow on a certain surface with a planar structure. A version of this construction is given below.

So, let $Q$ be a rational polygon. We set $\tilde{M} = Q \times R$ and introduce on $\tilde{M}$ the direct product topology (on $R$ we take the discrete topology). We say that two elements $(q_1, r_1)$ and $(q_2, r_2)$ in $\tilde{M}$ are equivalent if they are equal, or if $q_1 = q_2$ is a point on a side $a$ of $Q$ and $r_1^{-1}r_2 = a$, or if $q_1 = q_2$ is a vertex of $Q$ from which sides $a$ and $b$ issue. Let $M$ be the quotient, endowed with the topology which naturally embeds into $\tilde{M}$. For $r \in R$ we have $U_r = (\text{int} Q) \times \{r\}$ and $U_r$ can be extended to a planar structure on $Q$ with interior angle $\pi$ at the point $r \in R$ the neighbourhood $U_r$.

An arbitrary trajectory $\gamma$ on $\tilde{M}$ gives rise to $|R|/(2\pi)$ straight lines on $\gamma$.

The group $R$ acts on the phase space $\Phi = \Phi(Q)$, with the geodesic flow $\{T_t\}$ acting on the quotient space $\Phi/R$. We use the notation $\Phi(R)$.

First, the domain $D_1 \subset \Phi$ of $R$ on $\Phi$. Also, the points of $\tilde{D}$ act precisely as the phase space of the geodesic flow $\{T_t\}$ act inside $\Phi$ on the boundary $\partial D_1$. Consider a billiard flow in $Q$.

Further, let $J$ be an invariant surface of $\partial D_1$ and $\partial D_2 = M \times J$ is a fundamental domain of $R$. We note that $\partial D_2$ consists of two connected components. This implies that the phase space $\Phi$ splits into invariant surfaces and the corresponding foliation of the geodesic flow, which is transversal to these surfaces.

2.3. Interval exchange transformations. 

Definition 2.4. Suppose $a, a_1, \ldots, a_n, b$ in such a way that $a + a_1 + \ldots + a_n + b$ is a shift on each of the intervals $\Omega_i$.

Sometimes an interval exchange transformation is a bijection $f: \Omega \to \Omega$ with $f(a) = b$ and $f(a_1) = a_2, f(a_2) = a_3, \ldots, f(a_n) = a$.
Planar structures and billiards in rational polygons: the Veech alternative

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which sides \( a \) and \( b \) issue and \( r_1^{-1}r_2 \) belongs to the subgroup of \( R \) generated by
\( r_a \) and \( r_b \). Let \( M \) be the quotient space of \( M \) with respect to this equivalence
relation, endowed with the quotient topology. It can be readily seen that \( M \) is a
compact orientable surface, obtained by ‘gluing’ \( |R| \) copies of \( Q \). The polygon \( Q \)
naturally embeds into \( M \), as \( Q \times \{ \text{id} \} \). The collection of charts \( \{(U_r, f_r)\}_{r \in R} \), with
\( U_r = (\text{int } Q) \times \{r\} \) and \( f_r \) the map from \( U_r \) into \( \mathbb{R}^2 \) defined by \( f_r(q, r) = r(q) \),
can be extended to a planar structure \( \omega \) on \( M \) in a natural manner. The singular
points of this structure correspond to the vertices of \( Q \). In fact, let \( x \) be a vertex of
\( Q \) with interior angle \( \pi m/n \), where \( m/n \) is a fraction in its lowest terms. For each
\( r \in R \) the neighbourhood of \( (x, r) \in M \) consists of \( 2n \) copies of \( Q \) glued together,
and at this point \( \omega \) has a conical singularity with angle \( \pi m/n \cdot 2n = 2\pi m \). Thus, \( x \)
gives rise to \( |R|/(2n) \) singular points of multiplicity \( m \).

An arbitrary trajectory of the geodesic flow on \( M \) becomes a billiard flow under
the natural projection of \( M \) onto \( Q \). Conversely, each billiard trajectory in \( Q \) can
be straightened’ to a trajectory of the geodesic flow on \( M \).

The group \( R \) acts on \( M (\bar{\tau}(q, r) = (q, \bar{r}) \) ) and on the circle \( S^1 \), so it acts also
on the phase space \( \Phi = M \times S^1 \) of the geodesic flow on \( M \). This action commutes
with the geodesic flow \( \{T^t\} \), so there is a well-defined quotient flow \( \{T^t\}/R \) on the
quotient space \( \Phi/R \). We give two representations of this quotient flow.

First, the domain \( D_1 = \text{int } Q \times S^1 \subset \Phi \) is a fundamental domain for the action
of \( R \) on \( \Phi \). Also, the points on the boundary \( \partial D_1 \) which are identified under the
action of \( R \) are precisely the points of \( Q \times S^1 \) that are identified when constructing
the phase space of the billiard in \( Q \). Moreover, the billiard flow \( \{T^t_\Phi\} \) and the
geodesic flow \( \{T^t\} \) act identically on the elements of \( D_1 \) up to the time of hitting
the boundary \( \partial D_1 \). Consequently, the quotient flow \( \{T^t_\Phi\}/R \) is isomorphic to the
billiard flow in \( Q \).

Further, let \( J \) be an arc of the circle \( S^1 \) that is a fundamental domain for the
action of \( R \) on \( S^1 \). The end-points \( j_1 \) and \( j_2 \) of \( J \) are fixed points for certain
transformations \( r_1 \) and \( r_2 \) in \( R \) (\( r_1 \) and \( r_2 \) are reflections; they generate \( R \)). Then
\( D_2 = M \times J \) is a fundamental domain for the action of \( R \) on \( \Phi \). The boundary
\( \partial D_2 \) consists of two components, \( M \times \{j_1\} \) and \( M \times \{j_2\} \). Under the action of \( R \),
boundary points of the form \( (x, j_n) \) and \( (r(x), j_n) \), \( x \in M \), \( n = 1, 2 \), are identified.
This implies that the phase space of the quotient flow (or billiard) can be stratified
into invariant surfaces all except two of which are isomorphic to invariant surfaces
of the geodesic flow, while the two boundary surfaces can be two-sheeted covered
by such surfaces.

2.3. Interval exchange transformations.

Definition 2.4. Suppose we are given an interval \( I = [a, b] \) on the real number
axis and points \( a_1, \ldots, a_m \) in it with \( a < a_1 < \cdots < a_m < b \). An interval exchange
transformation is a bijective transformation \( T \) of \( I \setminus \{a, a_1, \ldots, a_m, b\} \) into \( I \) that is
a shift on each of the intervals \( (a, a_1), (a_1, a_2), \ldots, (a_m, b) \).

Sometimes an interval exchange transformation \( T \) is defined on some of the points
\( a, a_1, \ldots, a_m, b \) in such a way that it is left or right continuous at the corresponding
points and such that its bijectivity is not violated.
If $T$ is an interval exchange transformation, then all its powers $T^n$, $n \geq 1$, and its inverse $T^{-1}$ are also interval exchange transformations. For all $x \in I$ except for countably many of them, $T^n$ is defined for any $n \in \mathbb{Z}$. For an arbitrary interval exchange transformation $T$, we let $C(T)$ be the set of points $x \in I$ such that $T^n x$ and $T^{n_2} x$ are not defined for some $n_1 > 0$, $n_2 < 0$. Clearly, $C(T)$ is a finite set.

**Theorem 2.1** [15]. From the intervals into which $I$ is partitioned by the points of $C(T)$ we can, in a unique manner, form non-intersecting $T$-invariant sets $K_1, \ldots, K_s$, with $I \setminus C(T)$ as union, such that for each $i$, $1 \leq i \leq s$, either $K_i$ consists of intervals of the same length that are cyclically permuted by $T$ or the restriction of $T$ to $K_i$ is minimal, that is, the set $\{T^n x\}_{n \geq 0}$ is everywhere dense in $K_i$, for any $x \in K_i$, for which $T^n x$ is defined for all $n \geq 0$.

In the first situation we call $K_i$ a **periodic** component of $T$; in the second situation we call it a **minimal** component. The interval exchange transformation $T$ is said to be **minimal** if it has a single component which is also minimal. Theorem 2.1 implies the following sufficient condition for $T$ to be minimal.

**Theorem 2.2** [14]. If $C(T) = \emptyset$, then $T$ is either the identity or minimal.

Ergodic properties of interval exchange transformations are described in the following theorem.

**Theorem 2.3** [14]. Any aperiodic (that is, not having periodic components) interval exchange transformation has only finitely many ergodic invariant normalized Borel measures.

The relation between interval exchange transformations and planar structures is described in the following two constructions.

Let $M$ be a surface with a planar structure $\omega$, and let $\overline{v}$ be some direction. We consider an arbitrary geodesic interval $I$ perpendicular to $\overline{v}$ (I may contain interior singular points and, moreover, its beginning and end may coincide). Let $T$ be the first return map. It induces on $I$ a flow in the direction $\overline{v}$. The map $T$ is defined at $x \in I$ if the trajectory emitted from $x$ in the direction $\overline{v}$ intersects $I$ at a certain non-singular point $y$; in that case, $T x = y$.

**Proposition 2.4.** The map $T$ is an interval exchange transformation. Moreover, the number of intervals that are exchanged is bounded above by a constant depending on the planar structure only. Any trajectory in the direction $\overline{v}$ and emitted from a point $x \in I$ returns to $I$ or hits a singular point in a time span bounded above by a constant that is independent of $x$.

**Proof.** Let $x_1, \ldots, x_m$ be the points in $I$ such that the trajectory emitted from $x_i$ in the direction $\overline{v}$ hits a singular point and does not return to $I$. Clearly, the number of such points does not increase the sum of the multiplicities of all singular points of the planar structure. We add to them the (at most two) points that are mapped by $T$ to the end-points of $I$. Let $J$ be one of the subintervals into which the $x_1, \ldots, x_m$ partition $I$. Poincaré’s return theorem easily implies that the trajectories emitted from points of $J$ in the direction $\overline{v}$ return to $I$ and, moreover, do not increase $S/j$, where $S$ is the area of the surface $M$ and $j$ is the length of $J$.

Moreover, it is obvious that $D(T)$ has been proved complete.

**Definition 2.5.** A *saddle connection* of a planar structure is a saddle connection parallel to the same role as the points of $C(T)$ play in the transformation $T$.

The similar object for planar structures (two vertices) is called a *saddle vertex*.

**Proposition 2.5.** There exists a special flow under $T$ that preserves the saddle connections and constant on each of the saddles.

**Proof.** Let $I$ be a geodesic interval and $S$ the set of all trajectories emitted from $I$ by saddle connections parallel to $\overline{v}$; we use the proof of Proposition 2.4. Let $D(T)$ be a saddle connection and $D(T)$ leaves $D(I)$ invariant. We proceed to the exchange transformation on $I = \emptyset$. In the sequel, find intervals $I_1, \ldots, I_n$ such that $D(I_1), \ldots, D(I_m)$ are new points and saddle connections.

We first note that an interval may be partitioned in a pencil (or band) of points and saddles by saddle connections. We specify a point in a pencil (or band) of points and saddles by saddle connections. Pencil (or band) is a finite (it is at most the number of saddle connections) collection of trajectories in the direction $\overline{v}$.

Clearly, $D_i = D(I_i)$ for $1 \leq i \leq m$. We choose an interval $I$, which contains saddles and do not contain singular points, divide it into $m$ pencils (including saddles). We complement them with the saddles and point out the saddle connections.

The domains thus obtained are connected by saddle connections, and we can find arbitrarily many of them as required.

Propositions 2.4 and 2.5 imply the results of Theorems 2.1, 2.2 and 2.3.

**Theorem 2.6.** Let $D_i (D_i)$ be the saddle connections after deleting singular points from their domains. Then $D_i$ is invariant under the planar structure.
all its powers $T^n$, $n \geq 1$, and
ations. For all $x \in I$ except for $x \in \mathbb{Z}$. For an arbitrary interval
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e $M$ and $j$ is the length of $J$.

Moreover, it is obvious that the restriction of $T$ to $J$ is a shift. Thus, the assertion
has been proved completely.

Definition 2.5. A saddle connection of a planar structure is a geodesic interval
joining two singular points and not having singular points in its interior.

The phrase 'saddle connection' is related to the fact that a singular point of a
planar structure is a saddle point for the flow in a definite direction on $M$. Saddle
connections parallel to a direction $\bar{v}$ play, for the flow on $M$ in this direction, the
same role as the points of $C(T)$ play for the interval exchange transformation $T$.
The similar object for billiards in rational polygons (a billiard trajectory joining
two vertices) is called a generalized diagonal.

Proposition 2.5. The flow in a direction $\bar{v}$ on a surface $M$ can be represented as
a special flow under an interval exchange transformation with return time that is
constant on each of the exchange intervals.

Proof. Let $I$ be a geodesic interval on $M$ perpendicular to $\bar{v}$. Let $D(I)$ be the
union of all trajectories emitted from interior points of $I$ in the direction $\bar{v}$ or $-\bar{v}$. The
proof of Proposition 2.4 implies that $D(I)$ is a domain on $M$. It is bounded by
saddle connections and periodic trajectories parallel to $\bar{v}$. The flow in the direction
$\bar{v}$ leaves $D(I)$ invariant, and its restriction to $D(I)$ is a special flow over an interval
exchange transformation (on $I$); moreover, the return time is constant on each
exchanged interval. In view of the above, to prove the assertion it suffices to find
intervals $I_1, \ldots, I_m$, perpendicular to $\bar{v}$, such that the corresponding domains
$D(I_1), \ldots, D(I_m)$ are non-intersecting and have as union all of $M$ (up to singular
points and saddle connections parallel to $\bar{v}$).

We first note that an arbitrary periodic trajectory in the direction $\bar{v}$ is contained
in a pencil (or band) of periodic trajectories of a single period; this pencil is bounded
by saddle connections. Since the number of saddle connections in the direction $\bar{v}$
is finite (it is at most the sum of the multiplicities of the singular points) and each
saddle connection bounds at most two pencils, the number of pencils of periodic
trajectories in the direction $\bar{v}$ is finite as well. Let $D_1, \ldots, D_m$ be these pencils.
Clearly, $D_i = D(I_i)$ for certain intervals $I_1, \ldots, I_m$ perpendicular to $\bar{v}$. If the union
of these pencils (including their boundaries) is not $M$, then to complement them
we choose an interval $I_{m+1}$ perpendicular to $\bar{v}$. The domains $D(I_1), \ldots, D(I_{m+1})$
do not contain singular points. If the union of their closures is still not $M$, to
complement them we choose another interval $I_{m+2}$ perpendicular to $\bar{v}$, and so on.
The domains thus obtained are bounded by saddle connections parallel to $\bar{v}$ (and
each saddle connection bounds at most two domains), therefore there cannot be
arbitrarily many of them. Consequently, $M = \bigcup_{1 \leq i \leq m+n} D(I_i)$ for some $n \geq 0$, as
required.

Propositions 2.4 and 2.5 (and their proofs) make it possible to obtain analogues
of Theorems 2.1, 2.2 and 2.3 for flows.

Theorem 2.6. Let $D_1, \ldots, D_m$ be the domains into which the surface $M$ partitions
after deleting singular points and saddle connections parallel to $\bar{v} \in S^1$. Then each
$D_i$ is invariant under the flow on $M$ in the direction $\bar{v}$ and either it is a pencil of
periodic trajectories parallel to \( \vec{v} \) or the restriction of the flow to \( D_i \) is minimal. The number of domains is bounded by a constant not depending on the direction.

**Theorem 2.7.** If a planar structure \( \omega \) does not allow saddle connections parallel to a direction \( \vec{v} \), then either the flow on \( M \) in this direction is minimal or \( \omega \) does not have singular points and the whole surface \( M \) consists of a single pencil of periodic trajectories parallel to \( \vec{v} \).

**Theorem 2.8.** If the flow on a surface \( M \) in a direction \( \vec{v} \) is aperiodic, then there are only finitely many normalized Borel measures on \( M \) that are invariant and ergodic with respect to this flow.

We will now describe a second construction relating interval exchange transformations and planar structures. Let \( I = [a, b] \) be an interval on the real number axis, \( T \) an interval exchange transformation on \( I \), \( a_0 = a, a_1, \ldots, a_n = b \) the points of \( I \) at which \( T \) is not defined, and \( b_0 = a, b_1, \ldots, b_m = b \) the points at which \( T^{-1} \) is not defined. We put \( M = I \times [0, 1] \). By \( S \) we denote the subset of \( M \) consisting of the points \( (a_i, 1) \), \( 1 \leq i \leq n \), and \( (b_i, 0) \), \( 1 \leq i \leq m \). We identify intervals on the boundary of \( M \); we identify the points \( (a, t) \) and \( (b, t) \), \( t \in (0, 1) \), and we identify \( (x, 1) \) and \( (T_x, 0) \) for \( x \in I \) with \( T_x \) defined. Further, we identify the points of \( S \) that are identical end-points of identified intervals (that is, simultaneously upper or lower for vertical intervals, left or right for horizontal intervals). We denote by \( M \) the quotient space of \( M \) corresponding to the above identification. Then \( M \) is a compact orientable surface. The chart \( (U, \varphi) \) with \( U = (a, b) \times (0, 1) \) can be completed in a natural manner to a planar structure \( \omega \) on \( M \). The singular points of \( \omega \) correspond to the points of \( S \). By construction, the flow on \( M \) in the vertical direction is a special flow over \( T \) with return time equal to 1. In particular, this implies that the above-described construction cannot yield every planar structure.

There are other ways of associating a planar structure with an interval exchange transformation (see, for example, [4], §3).

### 2.4. Elementary planar structures.

**Theorem 2.9.** Let \( \omega \) be a planar structure on a surface \( M \), let \( m_1, \ldots, m_k \) be the multiplicities of its singular points, and let \( \chi \) be the Euler characteristic of \( M \). Then

\[
-\chi = \sum_{i=1}^{k} (m_i - 1).
\]

**Proof.** Both parts of this formula do not change when singular points are added or deleted, so we may assume that \( \omega \) has singular points.

**Lemma 2.10** [5]. If \( \omega \) has singular points, then every set of pairwise non-intersecting (that is, without common interior points) saddle connections can be complemented to a triangulation of \( M \) whose vertices are singular points, whose edges are saddle connections, and whose faces are triangles not containing singular points in their interior (\( \omega \)-triangles, see §6).

Let \( v, e, f \) be the number of connected components of \( \omega \) such that the formula \( \chi = v - e + f \) in the triangulation is equal to 0 in some integral, \( \gamma \), in which angles at the singular points are not equal to zero and \( -\chi = f/2 - k = \sum_{i=1}^{k} (m_i - 1) \).

In particular, the planar structures on \( \mathbb{S}^2 \) with 0 singularities only. On the other hand, they do not exist without singular points. Therefore all planar structures on \( \mathbb{S}^2 \) are isomorphic to every planar structure.

**Definition 2.6.** Let \( (U, \varphi) \) be a chart on \( M \). A homeomorphism \( f : M \to M \) is a planar structure if it maps the singular points in the local coordinates.

**Proposition 2.11.** An isomorphic planar torus to which are.

**Proof.** By Theorem 2.9 it is clear that if \( M \) is an isomorphic planar torus to which are.

A is in an arbitrary direction \( \vec{v} \) is a periodic trajectory. Perpendicular to \( \vec{v} \) intersects the geodesic that belong to be travelled along \( I \) from \( A \) to \( A' \) is periodic. This can be readily seen, the formula \(-s_1 \vec{v}_1 + s_2 \vec{v}_2 \) is periodic of periodic trajectories in \( A \) and \( A' \) of this interval belong to \( \omega_1 \) intersect \( \omega_2 \) just once. Let \( A_1 \) be a periodic trajectory of \( \omega_1 \) and \( \omega_2 \) intersect one another. It is isomorphic to the planar torus.

It is well known that \( \mathbb{Z} \vec{v}_1 \oplus \mathbb{Z} \vec{v}_2 \) are periodic,
Let \( v, e, f \) be the numbers of vertices, edges and faces of an arbitrary triangulation of \( M \) such as described in Lemma 2.10. Clearly, \( v = k, 3f = 2e \), so Euler’s formula \( \chi = v - e + f \) implies that \( -\chi = f/2 - k \). The sum of all angles of all faces in the triangulation is equal to \( \pi f \). On the other hand, it is equal to the sum of the full angles at the singular points of \( \omega \), that is, \( \sum_{i=1}^{k} 2\pi m_i \). Hence: \( f/2 = \sum_{i=1}^{k} m_i \) and \( -\chi = f/2 - k = \sum_{i=1}^{k} (m_i - 1) \), as required.

In particular, the proof of the theorem implies that there do not exist planar structures on the sphere. Planar structures on a torus can have removable singularities only. On the other hand, a planar structure on a surface of genus \( g > 1 \) must have at least one non-removable singularity.

Examples of planar structures on a torus are well known. Let \( \tilde{\omega}_1, \tilde{\omega}_2 \) be linearly independent vectors in \( \mathbb{R}^2 \). By \( T_{\tilde{\omega}_1, \tilde{\omega}_2} \) we denote the quotient space of the group \( \mathbb{R}^2 \) by the subgroup \( \mathbb{Z}\tilde{\omega}_1 \oplus \mathbb{Z}\tilde{\omega}_2 \). Then \( T_{\tilde{\omega}_1, \tilde{\omega}_2} \) is a torus, the canonical projection \( \pi: \mathbb{R}^2 \to T_{\tilde{\omega}_1, \tilde{\omega}_2} \) is a local homeomorphism, and the continuous maps from domains in \( T_{\tilde{\omega}_1, \tilde{\omega}_2} \) into \( \mathbb{R}^2 \) that are right inverse to \( \pi \) define on \( T_{\tilde{\omega}_1, \tilde{\omega}_2} \) a planar structure without singular points. This structure is called a planar torus. It will turn out that every planar structure on a torus can be obtained in this manner.

**Definition 2.6.** Let \( (M_1, \omega_1) \) and \( (M_2, \omega_2) \) be surfaces with planar structures. A homeomorphism \( f: M_1 \to M_2 \) is called an isomorphism of planar structures \( \omega_1 \) and \( \omega_2 \) if it maps the singular points of \( \omega_1 \) to the singular points of \( \omega_2 \) and is a shift in the local coordinates of \( \omega_1 \) and \( \omega_2 \).

**Proposition 2.11.** An arbitrary planar structure on a torus is isomorphic to a planar torus to which are added finitely many removable singular points.

**Proof.** By Theorem 2.9 it suffices to prove that a planar structure \( \omega \) without singular points on a torus \( M \) is isomorphic to a planar torus. We will first show that \( \omega \) has a periodic trajectory. Let \( A \in M \) and let \( I \) be a geodesic interval starting at \( A \) in an arbitrary direction \( \tilde{u}_1 \). The trajectory emitted from \( A \) in a direction \( \tilde{u}_2 \) perpendicular to \( \tilde{u}_1 \) intersects \( I \) at a point \( A' \). We denote by \( s_1, s_2 \) the distances to be travelled along \( I \) and along the trajectory, respectively, from \( A \) to \( A' \). As can be readily seen, the trajectory emitted from \( A \) in a direction \( \tilde{e}_1 \) parallel to \( -s_1 \tilde{u}_1 + s_2 \tilde{u}_2 \) is periodic. By Theorem 2.7, the whole surface \( M \) is a single pencil of periodic trajectories in the direction \( \tilde{e}_1 \), having the same length \( l_1 \). We draw the geodesic interval \( J \) in a direction \( \tilde{e}_2 \) perpendicular to \( \tilde{e}_1 \) whose length is the width \( w_1 \) of the pencil of periodic trajectories in the direction \( \tilde{e}_1 \). The end-points \( A \) and \( A' \) of this interval belong to the same trajectory of the pencil; all other trajectories intersect \( J \) just once. Let \( l_2 \) be the distance from \( A' \) to \( A \) when moving along the direction \( \tilde{e}_1 \). Then the trajectories parallel to \( \tilde{e}_2 = w_1 \tilde{e}_2 + l_2 \tilde{e}_1 \) form a pencil of periodic trajectories of length \( \|\tilde{e}_2\| \). The trajectories from the pencils parallel to \( \tilde{e}_1 \) and \( \tilde{e}_2 \) intersect one another just once. This implies that the planar structure \( \omega \) is isomorphic to the planar torus \( T_{\tilde{e}_1, \tilde{e}_2} \).

It is well known that the flows on \( T_{\tilde{e}_1, \tilde{e}_2} \) in directions parallel to vectors in \( \mathbb{Z}\tilde{e}_1 \oplus \mathbb{Z}\tilde{e}_2 \) are periodic, while the flows in all other directions are strongly ergodic.
(see, for example, [11]). Comparing this with Theorems 2.6 and 2.8, we conclude that this behavior is the simplest kind for the geodesic flow on a surface with a planar structure.

**Definition 2.7.** A planar structure $\omega$ on a surface $M$ is said to be *elementary* if the flow on $M$ in an arbitrary direction is either strongly ergodic or has only periodic components. (Here, parallel periodic trajectories from distinct pencils can have incommensurate lengths, that is, the flow in such a direction cannot, in general, be periodic.)

Each trajectory of the geodesic flow on $M$ corresponding to an elementary planar structure either hits a singular point, or is periodic, or is uniformly distributed with respect to the Lebesgue measure on $M$. Similarly, if the planar structure corresponding to a rational polygon $Q$ is elementary, then every bidirectional trajectory in $Q$ not hitting a vertex is either periodic or uniformly distributed inside $Q$.

Results from [9] imply that, in contrast to planar structures on a torus (which are all elementary), elementary planar structures on surfaces of genus $g > 1$ are rare: for a *typical* (see [9]) planar structure the set of directions whose flows are minimal but not strongly ergodic (or even non-ergodic with respect to the Lebesgue measure) has positive Hausdorff dimension. On the other hand, for an arbitrary planar structure, the flows in almost all directions are strongly ergodic (see §2.1). Moreover, there are directions whose flows have a periodic component (according to Masur [6], such directions densely fill $S^1$). An elementary proof of the existence of periodic trajectories for planar structures corresponding to rational polygons is due to Stépin (see [13]).

§3. The Veech alternative

In this section we will prove Veech's theorem [1], which gives a sufficient condition for a planar structure to be elementary. Basically, the proof follows the original lines proposed by Veech, but some individual steps are made simpler. Moreover, we will show that Masur's lemma (Lemma 3.5), the most important ingredient of the proof, follows from a theorem of Veech on interval exchange transformations.

3.1. The stabilizer of a planar structure. Let $\omega$ be a planar structure on a surface $M$, and $L$ a saddle connection of it. We can find a chart $(U, f)$ in the atlas of $\omega$ such that $U$ contains $L$ (without end-points). Then $f$ maps $L$ to an interval in $\mathbb{R}$. We orient this interval in the two possible ways, thus obtaining two oppositely pointing vectors, each of which we call a development of $L$. Clearly, a development of $L$ is well defined, that is, does not depend on the choice of the chart $(U, f)$.

We denote by $\text{SC}(\omega)$ the sequence whose terms are the developments of all saddle connections of $\omega$. If a vector $\bar{v} \in \mathbb{R}^2$ is a development of several saddle connections, it occurs in $\text{SC}(\omega)$ the corresponding number of times. The sequence $\text{SC}(\omega)$ is well defined up to the order of its terms.

**Proposition 3.1.** Suppose that $\omega$ has at least one singular point. Then the directions of the vectors in $\text{SC}(\omega)$ are everywhere dense on the unit circle. Simultaneously, the sequence $\text{SC}(\omega)$ does not have limit points.

---

**Proof.** For an arbitrary vector $\bar{v}$ and a vector $\bar{w}$ making with $\bar{v}$ an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$, let $AA'$ be a line segment with $A$ and $A'$ as end-points. We draw the trajectory of the geodesic flow on $M$ from $A$ to $A'$ measured along $L$ along the same direction forming an angle $\beta$ with the direction of $AA'$. For small $\beta$, the geodesic flow intersects $AA'$ after a time $t > 0$ from $A$. This situation is illustrated in Figure 3. Since $s_n \leq s$ and $\lim_{n \to \infty} s_n = s$ is required.

We will now prove that there is no saddle connection $L$ of $\omega$ with multiplicity of the singular points equal to $\pm 1$. Let $\bar{v} \in \mathbb{R}^2$, $\bar{v} \neq 0$. From the above, we know that there is some number of such interval exchanges, which imply the existence of a singular point of any such interval exchange before time $\delta$. The $\delta$-neighbourhoods of such points are not contained in any of the $\delta$-neighbourhoods of the singular points. As can be readily seen, the singular points are of order 1.

The above assertion on interval exchange transformations. We denote $\omega \cdot L$ the development of $L$.

If $\omega = \{(U_a, f_a)\}_{a \in \mathbb{Z}}$ is an interval exchange transformation on $L$, we denote it by $\omega$. In the case where $\omega$ is a development of a saddle connection of $\omega$ with the same length, we denote it by $\omega_{\bar{v}}$.

**Proposition 3.2.** The development of $\omega_{\bar{v}}$.

**Its restriction to the surface.**

**Proof.** Let $a, b \in \text{GL}(2, \mathbb{Z})$ be an interval exchange transformation of $\omega_{\bar{v}}$. Then $m(\bar{w}) = \frac{1}{\|ab^*\|}$, where $\|\cdot\|$ is the Euclidean norm. From this inequality, we obtain $m(\bar{w}) = \frac{1}{\|ab^*\|}$.
Proof. For an arbitrary $\bar{v} \in S^1$ and $\varepsilon > 0$ we have to prove that $SC(\omega)$ contains a vector making with $\bar{v}$ an angle less than $\varepsilon$. Let $A$ be a singular point of the planar structure, and let $AA'$ be a geodesic interval of length $s > 0$ perpendicular to $\bar{v}$. We draw the trajectory $L$ from $A$ in the direction $\bar{v}$. If it is a saddle connection there is nothing to prove. Otherwise $L$ intersects $AA'$ infinitely many times. Let $A_n$ be the point of $n$th intersection, and let $l_n$, $s_n$ be the distances from $A$ to $A'$ measured along $L$ and along the interval, respectively. We denote by $\bar{l}_n$ the direction forming an angle $\delta$ ($0 < \delta < \pi/2$) with $\bar{v}$ and an obtuse angle with the direction of $AA'$. For small $\delta$, the trajectory $L_\delta$ emitted from $A$ in the direction $\bar{l}_n$ intersects $AA'$ after a time $l_n/\cos \delta$ at the point $B_\delta$ lying at a distance $s_n - l_n \tan \delta$ from $A$. This situation persists for increasing $\delta$ until either $L_\delta$ suddenly becomes larger than $l_n/\cos \delta$ or $B_\delta$ merges with $A$ (for $\delta = \arctan(s_n/l_n)$). In any case we obtain a saddle connection forming with $\bar{v}$ an angle not exceeding $\arctan(s_n/l_n)$. Since $s_n \leq s$ and $\lim_{n \to \infty} l_n = +\infty$, we find that $\arctan(s_n/l_n) < \varepsilon$ for $n$ large, as required.

We will now prove that $SC(\omega)$ does not have limit points. Since the number of saddle connections with identical developments does not exceed the sum of the multiplicities of the singular points of $\omega$, it suffices to prove that $SC(\omega)$ does not have accumulation points. Each singular point has a neighbourhood not containing other singular points, so the length of an arbitrary element in $SC(\omega)$ is at least some $\varepsilon > 0$. Let $\bar{v} \in \mathbb{R}^2$, $\bar{v} \neq 0$. From all singular points we draw all possible geodesic intervals in the direction $\bar{v}$ of length $|\bar{v}|$ (the intervals may have interior singular points). The number of such intervals is finite (and depends on $\bar{v}$). Compactness considerations imply the existence of a $\delta > 0$ such that the trajectory emitted from an arbitrary point of any such interval in a direction perpendicular to $\bar{v}$ hits a singular point not before time $\delta$. Diminishing $\delta$, if necessary, we may also assert that the punctured $\delta$-neighbourhoods of the end-points of the drawn intervals do not contain singular points. As can be readily seen, the punctured neighbourhood of the vector $\bar{v}$ does not contain members of $SC(\omega)$. Since $\bar{v}$ is arbitrary, the assertion has been proved.

The above assertion implies, in particular, the existence of a shortest saddle connection. We denote its length by $m(\omega)$.

If $\omega = \{(U_\alpha, f_\alpha)\}_{\alpha \in A}$ is a planar structure on a surface $M$ and $a$ is a linear invertible operator in $\mathbb{R}^2$, then the atlas $\{(U_\alpha, a \circ f_\alpha)\}_{\alpha \in A}$ is also a planar structure on $M$; we denote it by $a\omega$. The planar structures $\omega$ and $a\omega$ have the same singular points with the same multiplicities, and also the same saddle connections. If $\bar{v}$ is a development of a saddle connection of $\omega$, then $a\bar{v}$ is a development of the same saddle connection with respect to $a\omega$. In particular, $SC(a\omega) = a(SC(\omega))$.

**Proposition 3.2.** The function $d: \text{GL}(2, \mathbb{R}) \to \mathbb{R}$, $d(a) = m(a\omega)$, is continuous. Its restriction to the subgroup $\text{SL}(2, \mathbb{R})$ is bounded.

**Proof.** Let $a, b \in \text{GL}(2, \mathbb{R})$ and let $\bar{v}$ be a development of the shortest saddle connection of $a\omega$. Then $(ba^{-1})\bar{v} \in SC(b\omega)$, whence

$$m(b\omega) \leq |(ba^{-1})\bar{v}| \leq |ba^{-1}| \cdot |\bar{v}| = |ba^{-1}| \cdot m(a\omega),$$

where $|| \cdot ||$ is the Euclidean operator norm. Interchanging the roles of $a$ and $b$ in this inequality, we obtain a second inequality:

$$||ab^{-1}||^{-1} \cdot m(a\omega) \leq m(b\omega) \leq ||ba^{-1}|| \cdot m(a\omega).$$
As $b \to a$, the quantities $\|ab^{-1}\|$ and $\|ba^{-1}\|$ tend to 1, therefore $m(b_\omega)$ tends to $m(a_\omega)$, that is, $d$ is a continuous function.

For any $a \in \text{SL}(2, \mathbb{R})$ the planar structures $a_\omega$ and $\omega$ give the same Lebesgue measure on the surface. Consequently, we will have proved that $d$ is bounded on $\text{SL}(2, \mathbb{R})$ if we can prove that $m(\omega) \leq \sqrt{2S}$, where $S$ is the area of the surface with planar structure $\omega$. Let $A$ be a singular point of $\omega$, and $\bar{v}$ an arbitrary direction. Starting at $A$ we draw the geodesic interval $I$ of length $l = \sqrt{S}$ in the direction $\bar{v}$. If $I$ hits a singular point, there is nothing to prove. If not, we draw from each point of the interval the trajectory in a direction $\bar{u}$ perpendicular to $\bar{v}$. After a time $t$ not exceeding $S/l = \sqrt{S}$, some such trajectory intersects $I$ again. If there is still no trajectory that has hit a singular point, then at time $t$ either some trajectory hits $A$ or $I$ intersects the trajectory emitted from $A$. Thus, if we draw from the points of $I$ the trajectories in the directions $\bar{u}$ and $-\bar{u}$, then at some time not exceeding $t$ some such trajectory hits a singular point $B$. Joining $A$ and $B$ we obtain a saddle connection whose projections on $\bar{v}$ and $\bar{u}$ do not exceed $\sqrt{S}$ and whose length is thus not greater than $\sqrt{2S}$. So $m(\omega) \leq \sqrt{2S}$, as required.

Definition 3.1. An affine automorphism of a planar structure $\omega$ is a homeomorphism $f : M \to M$ that maps singular points to singular points and is an affine map in the local coordinates of the atlas of $\omega$. This is equivalent to the fact that $f$ is an isomorphism of the planar structures $a_\omega$ and $\omega$, where $a \in \text{GL}(2, \mathbb{R})$. The operator $a$ is said to be the linear part of the automorphism $f$.

Since the areas of surfaces with isomorphic planar structures are equal, and since under transition from $\omega$ to $a_\omega$ surface area is multiplied by $|\det a|$, the determinant of the linear part of an affine automorphism must be equal to 1 or $-1$.

Definition 3.2. The stabilizer $\Gamma(\omega)$ of a planar structure $\omega$ is the set of operators $a \in \text{SL}(2, \mathbb{R})$ for which the planar structure $a_\omega$ is isomorphic to $\omega$.

In view of the above, the stabilizer consists of the linear parts of the affine automorphisms preserving the orientation of the surface.

Proposition 3.3. $\Gamma(\omega)$ is a discrete non-uniform subgroup of the group $\text{SL}(2, \mathbb{R})$.

Proof. If the planar structures $\omega_1$ and $\omega_2$ are isomorphic, then for any $a \in \text{GL}(2, \mathbb{R})$ the planar structures $a\omega_1$ and $a\omega_2$ are also isomorphic (the isomorphism is given by the same map). This implies that $\Gamma(\omega)$ is a group. Further, since $\text{SC}(a_\omega) = \text{SC}(\omega)$ for all $a \in \Gamma(\omega)$, Proposition 3.1 implies that $\Gamma(\omega)$ is a discrete group. Finally, let $\bar{v}$ be a development of an arbitrary saddle connection of the planar structure, and let $a$ be an operator in $\text{SL}(2, \mathbb{R})$ such that $a\bar{v} = 1/2\bar{v}$. Clearly, $m(a^n\omega) \to 0$ as $n \to \infty$. If $\Gamma(\omega)$ were a uniform subgroup, that is, the quotient group $\text{SL}(2, \mathbb{R})/\Gamma(\omega)$ were compact, then some subsequence $\{a^{n_i} \Gamma(\omega)\} \subset \text{SL}(2, \mathbb{R})/\Gamma(\omega)$ would converge to a certain $b\Gamma(\omega)$, with $b \in \text{SL}(2, \mathbb{R})$, as $i \to \infty$. Then there would be a sequence $\{\gamma_i\} \subset \Gamma(\omega)$ such that $a^{n_i} \gamma_i \to b$ in $\text{SL}(2, \mathbb{R})$ as $i \to \infty$. By Proposition 3.2, $m(a^{n_i} \gamma_i \omega) \to m(b\omega)$ as $i \to \infty$. This however, is impossible, since $m(a^{n_i} \gamma_i \omega) = m(a^{n_i} \omega)$ as $i \to \infty$ and $m(b\omega) \neq 0$. This contradiction shows that $\Gamma(\omega)$ is a non-uniform subgroup of $\text{SL}(2, \mathbb{R})$.

3.2. Veech's theorem

Theorem 3.4 (the Veech theorem). If $\gamma(\omega) = \gamma(\omega')$ is a full rank submodule, then its stabilizer $\Gamma(\omega)$ is a convex cocompact subgroup. Moreover, this holds if and only if $\alpha(\omega) = \alpha(\omega')$ for which the statement is true if and only if $\alpha(\omega') = \alpha(\omega')$.

The proof of this theorem involves several lemmas.

Lemma 3.5 (Masur [19] and Smillie [20]). Let $\Omega \subset \mathbb{H}$ be a set of singular points and is simply connected.

Then the flow in the vector field $v$ is defined on $\Omega$.

This formulation of the theorem is special for $\mathbb{H}$, however, it is completely equivalent.

Lemma 3.6. If the projection $\gamma(t) = g^t(1) \cdot \Gamma(\omega)$ is a non-constant map, $g^t(1) \cdot \Gamma(\omega)$ is a sequence of full rank submodules, and there are no singular points, then $\gamma(\omega)$ is a discrete cocompact subgroup of the modular group $\mathbb{H}$.

Proof. We proceed as in [10, 15]. By Proposition 3.3, the sequence $g^t(1) \cdot \Gamma(\omega)$ is a sequence of full rank submodules. Consider the sequence $\{\gamma_i\}$ such that $\gamma_i = g^{t_i}(1) \cdot \Gamma(\omega)$, where $t_i \to \infty$. Since $\gamma_i$ is a sequence of full rank submodules, this implies that $m(g^{t_i}(\omega)) \neq 0$ as $i \to \infty$. Then $m(g^{t_i}(h\omega)) \neq 0$ as $i \to \infty$. Then $m(h\omega) \neq 0$, we find that $\gamma(\omega)$ is a discrete cocompact subgroup of the modular group $\mathbb{H}$.

Lemma 3.7. Let $\|b\| > 1$ as $t \to \infty$. Then $\left\| g^t(\omega) \right\| \to \infty$ as $t \to \infty$.

Proof. The group $\text{SL}(2, \mathbb{R})$ acts on the infinite plane $\{z \in \mathbb{C} | \text{Im} z > 0\}$.

Moreover (see [16]), for any $D \subset \mathbb{H}$ with finitely many boundary elements, and the sides $a_i \in \Gamma$ of the form $\pm r_i \gamma_i$, where $\gamma_i$ is a vector (it acts on $\mathbb{H}$ by translation), there exists a sequence $a_i \in \Gamma$ of the form $\pm r_i \gamma_i$, where $\gamma_i$ is a vector (it acts on $\mathbb{H}$ by translation), there exists a sequence $a_i \in \Gamma$ of the form $\pm r_i \gamma_i$, where $\gamma_i$ is a vector (it acts on $\mathbb{H}$ by translation), there exists a sequence $a_i \in \Gamma$ of the form $\gamma_i(g^{-t}z)g^{-t}z$ (where $g \in \text{SL}(2, \mathbb{R})$).
3.2. Veech’s theorem.

**Theorem 3.4** (the Veech alternative). *Suppose that the planar structure \( \omega \) is such that its stabilizer \( \Gamma(\omega) \) is a lattice in \( \text{SL}(2, \mathbb{R}) \). Then \( \omega \) is an elementary planar structure. Moreover, the pencil in the direction \( \vec{v} \) has a single periodic component if and only if \( \alpha \vec{v} = \vec{v} \) for some \( \alpha \in \Gamma(\omega) \), \( \alpha \neq 1 \); if \( \omega \) has singular points, the latter is true if and only if \( \omega \) has a saddle connection parallel to \( \vec{v} \).

The proof of this theorem consists in the successive application of the following four lemmas.

**Lemma 3.5** (Masur [10], Theorem 1.1). *Suppose that the planar structure \( \omega \) has singular points and is such that \( m(g^t \omega) \to 0 \) as \( t \to +\infty \), where

\[
 g^t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in \text{SL}(2, \mathbb{R}).
\]

Then the flow in the vertical direction is strongly ergodic.

(This formulation of the lemma is different from the original formulation; however, it is completely equivalent to it.)

**Lemma 3.6.** *If the planar structure \( \omega \) is such that \( m(g^t \omega) \to 0 \) as \( t \to +\infty \), then \( g^t(1 \cdot \Gamma(\omega)) \to \infty \) in \( \text{SL}(2, \mathbb{R})/\Gamma(\omega) \) as \( t \to +\infty \).

**Proof.** We proceed as in the proof of Proposition 3.3. Suppose that \( g^t(1 \cdot \Gamma(\omega)) \to \infty \) as \( t \to +\infty \). Then for some sequence \( \{t_i\} \subset \mathbb{R} \) tending to \( +\infty \) we have \( g^{t_i}(1 \cdot \Gamma(\omega)) \to h \cdot \Gamma(\omega) \), where \( h \in \text{SL}(2, \mathbb{R}) \). Furthermore, we can find a sequence \( \{\gamma_i\} \subset \Gamma(\omega) \) such that \( g^{t_i} \gamma_i \to h \) in \( \text{SL}(2, \mathbb{R}) \) as \( i \to \infty \). By Proposition 3.2, this implies that \( m(g^{t_i} \gamma_i \omega) \to m(h \omega) \) as \( i \to \infty \). Since \( m(g^{t_i} \gamma_i \omega) = m(g^{t_i} \omega) \) and \( m(h \omega) \neq 0 \), we find that \( m(g^{t_i} \omega) \to 0 \) as \( t \to +\infty \). This proves the lemma.

**Lemma 3.7.** *Let \( \Gamma \) be a lattice in \( \text{SL}(2, \mathbb{R}) \) such that \( g^t(1 \cdot \Gamma) \to \infty \) in \( \text{SL}(2, \mathbb{R})/\Gamma \) as \( t \to +\infty \). Then \( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma \) for some \( \alpha \neq 0 \).

**Proof.** The group \( \text{SL}(2, \mathbb{R}) \) acts on the Lobachevskii plane \( \mathbb{H}^2 \), realized as the half-plane \( \{z \in \mathbb{C} | \text{Im} z > 0\} \), by linear-fractional transformations:

\[
\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) z = \frac{\alpha z + \beta}{\gamma z + \delta}.
\]

Moreover (see [16]), for the action of the lattice \( \Gamma \) there is a fundamental polygon \( D \subset \mathbb{H}^2 \) with finitely many sides. \( D \) has finitely many vertices \( A_1, \ldots, A_k \) on the absolute, and the sides of \( D \) through \( A_i \) are mapped to each other by an operator \( a_i \in \Gamma \) of the form \( \pm r_i \sigma r_i^{-1} \), where \( \sigma \) is the operator formed by a fixed horizontal vector (it acts on \( \mathbb{H}^2 \) by the shift \( z \to z + \alpha \)) and \( r_i \) is the rotation operator mapping \( \infty \) to \( A_i \).

For each \( z \in \mathbb{H}^2 \) we let \( \gamma(z) \) be the element of \( \Gamma \) for which \( \gamma(z)z \in D \). The curve \( \gamma(g^{-t}z)g^{-t}z \) (which is, in general, discontinuous) does not have limit points
as \( t \to +\infty \). In fact, otherwise \( \gamma(g^{-i}z)g^{-i}z \to \tilde{z} \in \mathbb{H}^2 \) as \( i \to \infty \), where \( \{t_i\} \) is a sequence tending to \( +\infty \). Since the operators mapping \( z \) into a closed neighborhood of \( \tilde{z} \) form a compact set in \( SL(2, \mathbb{R}) \), the sequence \( \{\gamma(g^{-i}z)g^{-i}z\} \) would have a limit point in \( SL(2, \mathbb{R}) \). Hence, the sequence of inverse elements \( g^{1-}\gamma(g^{-i}z)g^{1-}\) would also have a limit point, contradicting the fact that \( g^{1}\to +\infty \) in \( SL(2, \mathbb{R})/\Gamma \) as \( t \to +\infty \).

We put \( \Pi(c) = \{z \in \mathbb{C} \mid \Im z > c \geq 0\} \), \( \Pi(c; \alpha, \beta) = \{z \in \Pi(c) \mid \alpha \leq \Re z \leq \beta\} \), \( \Pi(c; \alpha, \beta) = \{z \in \Pi(c; \alpha, \beta) \mid \Im z = c\} \). The polygon \( D \) consists of a compact part \( K \) and \( k \) beaks (or wedges) \( r_i(\Pi(c; \alpha_i, \beta_i)) \) (here, \( \alpha_i \) and \( \beta_i \) are uniquely determined by \( D \)). For every \( z \in r_i(\Pi(c; \alpha_i, \beta_i)) \), \( \gamma(z) \) is easily seen to be a power of \( a_{\alpha} \), and \( \gamma(z) \in r_i(\Pi(c; \alpha_i, \beta_i)) \).

If \( z \in r_i(\partial \Pi(c; \alpha_i, \beta_i)) \), then \( \gamma(z) \in r_i(\Pi(c; \alpha_i, \beta_i)) \).

We fix a point \( z \in \mathbb{H}^2 \). Since the curve \( \gamma(g^{-t}z)g^{-t}z \) does not have limit points as \( t \to +\infty \), it does not intersect \( K \) for \( t \) larger than some \( t_0 \). We put \( \bar{z} = \gamma(g^{-t_0}z) \); then \( \bar{z} = r_i(\Pi(c; \alpha_i, \beta_i)) \), where \( 1 \leq i \leq k \). The curve \( g^{-t}z \) is a Euclidean straight line, tending, as \( t \to +\infty \), to the point 0 on the absolute. Therefore the curve \( \gamma(g^{-t}z) \), \( t \geq t_0 \), is a Euclidean straight line or circle tending to the point \( \bar{z} \). If \( \gamma(0) \neq A_i \), then at a certain moment of time \( t_1 \geq t_0 \) the curve \( \gamma(g^{-t}z) \) would intersect the boundary of the domain \( r_i(\Pi(c; \alpha_i, \beta_i)) \); at the same time the curve \( \gamma(g^{-t}z)g^{-t}z \) would intersect the arc \( r_i(\Pi(c; \alpha_i, \beta_i)) \) and would enter \( E \). This contradicts the choice of \( t_0 \). So \( \gamma(0) = A_i \), and then \( \alpha = \gamma^{-1}a_{\alpha} \gamma \in \Gamma \) maps the point 0 to itself. Since \( \bar{z} \) is conjugate to \( a_{\alpha} \), and, consequently, also to some \( \pm \bar{z} \), it must have the form \( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \), \( \alpha \neq 0 \). Moreover, \( \bar{a}^2 = \begin{pmatrix} 1 & 0 \\ 2\alpha & 1 \end{pmatrix} \in \Gamma \). This proves the lemma.

**Lemma 3.8.** If \( a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma(\omega) \), \( \alpha \neq 0 \), and the planar structure \( \omega \) has singular points, then the flow in the vertical direction corresponding to \( \omega \) has only periodic components. Moreover, for each pencil of periodic trajectories, the ratio of the length to the width commensurate with \( \alpha \).

**Proof.** An affine automorphism with linear part \( a \) acts as a permutation on the finite set \( \{L_1, \ldots, L_k\} \) consisting of the trajectories emitted from singular points of \( \omega \) in the vertical direction (upwards and downwards). Hence, for some \( n \in \mathbb{N} \) the affine automorphism \( \varphi^n \) maps each trajectory \( L_i \) to itself. Moreover, clearly, all points on the trajectory remain fixed. Consequently, the closure of the trajectory does not have interior points (otherwise the linear part of \( \varphi^n \) would be the identity). By Theorem 2.6, \( L_i \) is a saddle connection. Now let \( p \) be a point on the surface not lying on a vertical saddle connection and \( L \) the vertical trajectory passing through \( p \). For some \( \varepsilon > 0 \) the point \( p \) does not belong to the \( \varepsilon \)-neighborhood \( U^p \) of the union of the saddle connections \( L_1, \ldots, L_k \). However, it follows from the above that \( U^p \) is invariant under the flow in the vertical direction, so no point of \( L \) can belong to \( U^p \). Applying Theorem 2.6 we find that \( L \) is a periodic trajectory.

So, the flow in the vertical direction splits into periodic pencils. Moreover, under the action of \( \varphi^n \) a point in the pencil at a distance \( x \) from its left boundary moves vertically upwards through a distance \( x \cdot n\alpha \). Since \( \varphi^n \) is continuous and leaves
\( \mathbb{R}^2 \) as \( i \to \infty \), where \( \{t_i\} \) is a
sing \( z \) into a closed neighbourhood \( \{\gamma(g^{-t_i}z)g^{-t_i}\} \) would have
verse elements \( \{g^{t_i}(\gamma(g^{-t_i}z))^{-1}\} \)
hat \( g'(1-\Gamma) \to \infty \) in \( \text{SL}(2,\mathbb{R})/\Gamma \)
\( \mathcal{D} = \{z \in \Pi(c) \mid \alpha \leq \text{Re} z \leq \beta\} \),
ygon \( D \) consists of a compact \( \leq i \leq k \), separated from \( K \) by
determined by \( D \). For every \( i \), and \( \gamma(z) \in r_i(\Pi(c_i; \alpha_i, \beta_i)) \).

\( \gamma(z) \) does not have limit points
some \( t_0 \). We put \( \tilde{\gamma} = \gamma(g^{-t_0}z) \);
The curve \( g^{-t_i}z \) is a Euclidean \( \gamma \) on the absolute. Therefore
line or circle tending to the
at of time \( t_1 \geq t_0 \) the curve \( n r_i(\Pi(c_i)) \); at the same time
\( c_i; \alpha_i, \beta_i \) and would enter \( K \).
then \( \tilde{a} = \tilde{\gamma}^{-1}_1 a \in \Gamma \) maps the
sequentially, also to some \( \pm \alpha \),
\( x, \tilde{a}^2 = \begin{pmatrix} 1 & 0 \\ 2\alpha & 1 \end{pmatrix} \in \Gamma \). This

\( \mu \) the planar structure \( \omega \) has
n corresponding to \( \omega \) has only
periodic trajectories, the ratio of
acts as a permutation on the
mented from singular points of
itself. Moreover, clearly, all
the closure of the trajectory
of \( \varphi^n \) would be the identity).
be a point on the surface not
trajectory passing through \( p \).
ighbourhood \( U_p \) of the union
flows from the above that \( \varphi^n \),
so no point of \( L \) can belong
odic trajectories. Moreover, under
from its left boundary moves
\( \varphi^n \) is continuous and leaves
invariant the points on the boundary of the pencil, for a value of \( x \) equal to the
width \( w \) of the pencil the magnitude of the motion must be a multiple of the
length \( l \) of the pencil. Thus, \( w \cdot n \alpha = l m, m \in \mathbb{N} \); in particular, \( \alpha \) and \( l/w \) are
commensurate.

**Proof of Theorem 3.4.** First, since the group of automorphisms (shifts) of the planar
torus acts transitively on it, by adding a removable singularity to the planar torus
its stabilizer does not change. By Proposition 2.11 we may assume that, without
loss of generality, the planar structure \( \omega \) does have singular points.

Let \( \bar{v} \) be an arbitrary direction and \( b \) the rotation mapping \( \bar{v} \) into the vertical
direction. Clearly, \( \Gamma(b_\omega) = b_\Gamma(\omega)b^{-1} \), hence \( \Gamma(b_\omega) \) is also a lattice. Furthermore,
the planar structures \( \omega \) and \( b_\omega \) induce the same Lebesgue measure on the surface,
and the flow in the vertical direction for \( b_\omega \) coincides with the flow in the direction
\( \bar{v} \) for \( \omega \). Finally, \( SC(b_\omega) \) contains a vertical vector if and only if \( SC(\omega) \) contains
a vector parallel to \( \bar{v} \), and if \( a\bar{v} = \bar{v} \), with \( a \in \Gamma(\omega) \), \( a \neq 1 \), then \( a_1(bv) = bv \), with
\( a_1 = bab^{-1} \in \Gamma(b_\omega) \), \( a_1 \neq 1 \). Thus, without loss of generality we may assume that
the direction \( \bar{v} \) mentioned in the conditions of the theorem is vertical. If \( \omega \) has a
vertical saddle connection, then clearly \( m(g') \omega \to 0 \) as \( t \to +\infty \) and, by Lemmas 3.6
and 3.7, \( \Gamma(\omega) \) must contain a non-identity element mapping the vertical vector
to itself. In the presence of such an element, the flow in the vertical direction can have
periodic components only, by Lemma 3.8. Finally, if the flow in the vertical
direction has a periodic component, then a vertical saddle connection can be found
on the boundary of the component. So, to finish the proof it remains to show that
the flow in the vertical direction is strongly ergodic if it does not split into periodic
components. This follows from Lemmas 3.5–3.8.

We finish this subsection with another lemma, which is in a certain sense the
converse of Lemma 3.8 (we will need it in §4).

**Definition 3.3.** The least common multiple \( \text{LCM}(r_1, \ldots, r_k) \) of positive rationally
commensurate numbers \( r_1, \ldots, r_k \) is the least positive integer that is an integral
multiple of each of these numbers.

**Lemma 3.9.** Suppose that the flow in the vertical direction corresponding to the
planar structure \( \omega \) has only periodic components. If \( r_1, \ldots, r_k \), the ratios of the
length to the width of all pencils in the flow, are commensurate, then
\( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma(\omega) \),
where \( \alpha = \text{LCM}(r_1, \ldots, r_k) \).

**Proof.** We define a bijective map \( \varphi \) from the surface onto itself as follows: \( \varphi \) leaves
invariant each point \( p \) not belonging to the vertical saddle connection; otherwise \( p \)
is moved vertically upwards through a distance \( x \cdot \alpha \), where \( x \) is the distance from
\( p \) to the left boundary of the pencil of vertical periodic trajectories containing \( p \).
Since \( \alpha \) is an integral multiple of each of the numbers \( r_1, \ldots, r_k \), it follows that \( \varphi \)
is a homeomorphism. By construction, \( \varphi \) is then an affine automorphism of \( \omega \) with
linear part \( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \).
3.3. Masur’s lemma. Let $T$ be an interval exchange transformation on the interval $I = [a,b]$ (see §2.3) and $a_0 = a, a_1, \ldots, a_k = b$ the points in $I$ at which $T$ is not defined. We denote by $\varepsilon(T)$ the length of the shortest intervals into which $I$ is partitioned by these points $a_0, a_1, \ldots, a_k$. Furthermore, for each $n \in \mathbb{N}$ we put $\varepsilon_n(T) = \varepsilon(T^n)$.

**Theorem 3.10** [3], [15]. If $n\varepsilon_n(T) \to 0$ as $n \to \infty$ and $T$ is minimal, then $T$ is strongly ergodic.

Now let $\omega$ be the planar structure on the surface $M$, and let $\nu$ be an arbitrary direction. For each $l > 0$ we consider its saddle connections of length at most $l$ and not parallel to $\nu$, and project these onto the direction orthogonal to $\nu$. We denote by $E_l(\nu)$ the shortest of these projections.

**Theorem 3.11.** If $lE_l(\nu) \to 0$ as $l \to +\infty$ and the flow on $M$ in the direction $\nu$ is minimal, then this flow is strongly ergodic.

This theorem follows from the following proposition.

**Proposition 3.12.** Theorems 3.10 and 3.11 are equivalent. Lemma 3.5 follows from Theorem 3.11.

**Proof.** We will first prove that Theorem 3.11 follows from Theorem 3.10. We suppose that the flow on $M$ in the direction $\nu$ is minimal and that $lE_l(\nu) \to 0$ as $l \to +\infty$. We consider an arbitrary geodesic interval $I$ perpendicular to $\nu$. By Proposition 2.4 and Theorem 2.6, the flow in the direction $\nu$ induces on $I$ a minimal interval exchange transformation $T$ and is a special flow over $T$. The set $S$ of points of $I$ from which a trajectory issues in the direction $\nu$ and hits a singular point, is nowhere dense in $I$ (this follows from Theorem 2.6). By making $I$ smaller we may assume that its end-points are contained in $S$. Let $S_1$ be the set of points of $S$ for which the trajectory in the direction $\nu$ hits a singular point before the first return to $I$, returns to an end-point of $I$, or is an end-point of $I$ itself. The set $S_1$ is finite, and $I \setminus S_1$ consists of finitely many intervals on each of which $T$ acts as a shift. Consequently, we may assume that $T$ is not defined at the points of $S_1$. Moreover, the trajectory in the direction $\nu$ emitted from a point $x \in I$ passes through a singular point before a time $t_1$ if $x \notin S_1$, and returns to $I$ before a time $t_2$ if $x \notin S_1$ (with $t_1, t_2$ certain constants).

Let $n \in \mathbb{N}$, and let $x_1, x_2$ be points of $I$ at which $T^n$ is not defined and with mutual distance $\varepsilon_n(T)$. The trajectories in the direction $\nu$ emitted from $x_1$ and $x_2$ hit singular points lying at distance not exceeding $(n-1)t_2 + t_1$. This implies the existence of a saddle connection whose projection onto the direction $\nu$ has length at most $(n-1)t_2 + t_1$, while its projection on the orthogonal direction is positive and has length at most $\varepsilon_n(T)$. The length of this saddle connection does not exceed $2nt_2$ for $n$ large. Consequently, $E_{2nt_2}(\nu) \leq \varepsilon_n(T)$ for $n$ large. So, $n\varepsilon_n(T) \to 0$ as $n \to \infty$, and $T$ is strongly ergodic by Theorem 3.10. In this case the flow on $M$ in the direction $\nu$ is also strongly ergodic, being a special flow over $T$.

Now we will, conversely, derive Theorem 3.10 from Theorem 3.11. Let $T$ be a minimal interval exchange transformation on $I$ and assume that $n\varepsilon_n(T) \to 0$ as $n \to \infty$. We associate with $T$ a planar structure on a certain surface $M$, as described in §2.3. The interval $I$ embeds into $M$ as a horizontal interval, and the flow on $M$ in the vertical direction fix an $n \in \mathbb{N}$ and consider a horizontal projection $g'$ of $M$ in such a way that the direction $\nu$ is a singular point lying at an arbitrary point of $M$, and $\varepsilon_n(T)$ is at least $\delta$. Hence $\varepsilon_n(T)$ is at least $\delta$ as $l \to +\infty$. By Theorem 3.11, $T$ is strongly ergodic, and hence its generating map.

Finally, we will show that $\delta$ is a generating number for $T$. The condition $m(g'(\omega)) \to 0$ as $\delta > 0$ and sufficiently large, and $\omega$ has length at most $\delta$, where $\omega$ is also equivalent to the flow on $M$. Thus the condition is also equivalent to the flow on $M$. The projection $g'(\omega)$ is at most $\delta^2/l$, and $g'$ differs from $T$, the choice of the saddle connection $\omega$, and $g'(g')(\omega)$, $m(g'(\omega)) \to 0$ as $l \to +\infty$.

It follows from the above that, if $\delta > 0$ and sufficiently large, $m(g'(\omega)) \to 0$ as $l \to +\infty$. Hence the planar structure $M$ is minimal, and $\omega$ is a generating number for the flow on $M$. Therefore Theorem 3.11 hold.

3.4. Pencils of periodic trajectories. Let $M$ be a surface of genus $g$ having singular points each of the form $x + y$ from a singular point $x \in M$. The points $x$ and $y$ are the singular points of $I$ and $g'(\omega)$ is a singular point of $g'(\omega)$. The trajectories $x + y$ are bounded by saddle connections $\omega$ and $\omega'$, and there will be several pencils corresponding to $\omega$ multiple if its trajectories at least $\omega$ times.

For each $R > 0$ we denote by $\Omega_R$ the $R$-neighborhood of the region of the pencils of periodic trajectories $\omega$. Let $N_R(\omega)$ denote the number of pencils corresponding to $\omega$ in the $R$-neighborhood of $\omega$. For large $R$:

$$c_\omega \cdot R^2 \leq N_R(\omega)$$
xchange transformation on the = h the points in I at which T c shortest intervals into which I x more, for each n ∈ N we put
∞ and T is minimal, then T is
x M, and let ṽ be an arbitrary
sections of length at most l and
ion orthogonal to ṽ. We denote
the flow on M in the direction ṽ
tion.

equivalent. Lemma 3.5 follows
ows from Theorem 3.10. We
minimal and that IE_q(ṽ) → 0
terval I perpendicular to ṽ. By
ction ṽ induces on I a minimal
ow over T. The set S of points
v and hits a singular point, is
2.6). By making l smaller we
S. Let S be the set of points
its a singular point before the
s an end-point of I itself. The
y intervals on each of which T
T is not defined at the points
itten from a point x ∈ I passes
and returns to I before a time
ich T^n is not defined and with
ction ṽ emitted from x_1 and x_2
(n-1)/2 + t_1. This implies the
onto the direction ṽ has length
horizontal direction is positive
nle connection does not exceed
for n large. So, ε_h(T) → 0 as
). In this case the flow on M in
icial flow over T.
from Theorem 3.11. Let T be
and assume that nε_h(T) → 0
are on a certain surface M, as
as a horizontal interval, and the
flow on M in the vertical direction ṽ is a special flow over T, hence minimal. We
fix an n ∈ N and consider a saddle connection L of length not exceeding n and
with horizontal projection E_n(ṽ). We denote the end-points of L by A_1 and A_2,
in such a way that the direction from A_1 to A_2 makes an acute angle with ṽ. The
trajectory emitted from A_1 in the direction ṽ hits after time t a point x_1 ∈ I
at which T is not defined. Let x_2 ∈ I be the point of I nearest to x_1 with the
property that the trajectory emitted from it in the direction ṽ passes through a
singular point lying at a distance not exceeding n + 1. Clearly, T^n is not defined
at x_2. Hence ε_h(T) is at most equal to the distance between x_1 and x_2, which, by
the choice of L, does not exceed E_n(ṽ). So, ε_h(T) ≤ E_n(ṽ), that is, IE_q(ṽ) → 0
as t → +∞. By Theorem 3.11, the flow in the direction ṽ is strongly ergodic, and
hence its generating map T is strongly ergodic.

Finally, we will show that Lemma 3.5 is a consequence of Theorem 3.11. Clearly,
the condition m((y'ω) → 0 as t → +∞ is equivalent to the fact that for an arbitrary
δ > 0 and sufficiently large l there is a saddle connection whose vertical projection
has length at most δl, while the length of its horizontal projection is at most δ/l.
It is also equivalent to the fact that for sufficiently large l there is a saddle connection
whose vertical projection has length at most c_δ l, while the length of its horizontal
projection is at most δ^2 / l. The condition lim_{t→+∞} IE_q(ṽ) = 0, where ṽ is the vertical
direction, differs from the previous condition only by an additional restriction in
the choice of the saddle connections: they cannot be vertical. Thus, the condition
m((y'ω) → 0 as t → +∞ implies that IE_q(ṽ) → 0 as t → +∞. Moreover, in
this case the planar structure ω does not have vertical saddle connections and, by
Theorem 2.7, the flow in the vertical direction is minimal. Thus, all conditions in
Theorem 3.11 hold.

It follows from the above proof that, for a flow on M in the direction ṽ,
Theorem 3.11 asserts more than Lemma 3.5 if and only if ω allows saddle
connections parallel to ṽ.

3.4. Pencils of periodic trajectories. Let ω be a planar structure on a sur-
face M, having singular points. We assume that the geodesic trajectory emitted
from a singular point x ∈ M in a direction ṽ ∈ S is a domain bounded by saddle
connections in the direction ṽ (see §2.3). Moreover, there can be several pencils
corresponding to the same development. A pencil is called multiple if its trajectories are periodic trajectories of minimal length, traversed
several times.

For each R > 0 we denote by N(R) and S(R) the number and the sum of the
areas of the pencils of periodic trajectories of length at most R, without counting
multiple pencils. Let N^*(R) and S^*(R) be the same quantities, but including
multiple pencils in the count. Moreover, we denote by N_0(R) the number of saddle
connections of ω whose lengths are at most R. Masur [7], [8] has shown that for
large R:

ε_0 · R^2 ≤ N_0(R) ≤ ε_1 · R^2,
ε_0 · N_0(R) ≤ N(R) ≤ N_0(R),
where $c_+, c_-, c_0$ are positive constants. Since $N^*(R) = N(R) + N(R/2) + N(R/3) + \cdots$, for large $R$ we also have

$$c_+^* \cdot R^2 \leq N^*(R) \leq c_-^* \cdot R^2,$$

where $c_+, c_-^*$ are positive constants. Moreover, we clearly have $S(R) \leq S \cdot N(R)$, $S^*(R) \leq S \cdot N^*(R)$, where $S$ is the surface area of $M$.

The following theorem supplements Veech's result on the distribution of periodic trajectories of planar structures, obtained in [1].

**Theorem 3.13.** If the stabilizer $\Gamma(\omega)$ is a lattice, then each of the quantities $N_0(R), N(R), S(R), N^*(R), S^*(R)$ has, as $R \to \infty$, asymptotic behaviour of the form $cR^2 + o(R^2)$, where $c$ is a positive constant.

**Proof.** Since $\Gamma(\omega)$ is a lattice, we can find finitely many operators $a_1, \ldots, a_k \in \Gamma(\omega)$ and non-zero vectors $\vec{v}_1, \ldots, \vec{v}_k$, where $a_i \vec{v}_i = \pm \vec{v}_i$ but $a_i \neq \pm 1$, such that every operator $a \in \Gamma(\omega)$, $a \neq \pm 1$, having eigenvector $\vec{v}$ with eigenvalues $\pm 1$ is conjugate in $\Gamma(\omega)$ to an operator of the form $\pm a_i^\gamma$, $1 \leq i \leq k$, $\gamma \neq 0$ (see [16]). Moreover, $\vec{v}$ is collinear with a vector of the form $\gamma \vec{v}_i$, $\gamma \in \Gamma(\omega)$. Without loss of generality we may assume that the vector $\vec{v}_i$ in this representation is uniquely determined. By Theorem 3.4, the vectors parallel to saddle connections (or periodic trajectories) of $\omega$ are precisely the fixed vectors of the non-identity operators in $\Gamma(\omega)$, that is, vectors of the form $\gamma \vec{v}_i$, $1 \leq i \leq k$, $\gamma \in \Gamma(\omega)$, and vectors collinear with these. Let $l_1^*, \ldots, l_m^*$ be the lengths of the saddle connections parallel to $\vec{v}_i$, let $L_1^*, \ldots, L_m^*$ be the lengths of the pencils of periodic trajectories parallel to $\vec{v}_i$, and let $S_1^*, \ldots, S_m^*$ be the areas of these pencils. If $\gamma \in \Gamma(\omega)$ and $\varphi$ is an affine automorphism of $\omega$ with linear part $\gamma$, then $\varphi$ is a bijective correspondence between the saddle connections and the pencils of periodic trajectories parallel to $\vec{v}_i$ and $\gamma \vec{v}_i$, respectively. Under this correspondence, the lengths of the saddle connections and pencils are multiplied by $|\gamma \vec{v}_i|/|\vec{v}_i|$, while the areas of the pencils are not changed. Thus, from what was said above we obtain

$$N_0(R) = \sum_{i=1}^k \sum_{j=1}^{n_i} N_{\vec{v}_i}(R, |\vec{v}_i|/l_i), \quad N(R) = \sum_{i=1}^k \sum_{j=1}^{n_i} N_{\vec{v}_i}(R, |\vec{v}_i|/L_i),$$

$$S(R) = \sum_{i=1}^k \sum_{j=1}^{n_i} S_j^*,\quad N_{\vec{v}_i}(R, |\vec{v}_i|/L_i),$$

where $N_{\vec{v}_i}(R)$ is the number of vectors of the form $\gamma \vec{v}_i$, $\gamma \in \Gamma(\omega)$, of length at most $R$, considered up to multiplication by $\pm 1$. We now use the following lemma, which was proved by Veech.

**Lemma 3.14** [1]. Let $\vec{v}$ be a non-zero vector which is fixed under a non-identity element of the lattice $\Gamma \subseteq \text{SL}(2, \mathbb{R})$, and let $\vec{u}$ be a vector of length $|\vec{v}|^{-1}$ orthogonal to $\vec{v}$. Let $\alpha$ be the minimum of the quantities $\langle a \vec{u}, \vec{v} \rangle / (\langle \vec{v}, \vec{v} \rangle)$ over all $a \in \Gamma$, $a \neq \pm 1$,

such that $a \vec{v} = \pm \vec{v}$. Then

$$N_{\vec{v}}(R) = \frac{1}{2\pi} \int_{R \cdot \mathbb{R}} \text{Vol}(H^2, \Gamma) d\varphi,$$

where $\text{Vol}(H^2, \Gamma)$ is the non-Euclidean volume on $\mathbb{H}^2$.

This lemma, plus the formula for the quantities $N_0(R), N(R), S(R), N^*(R), S^*(R)$ has, as $R \to \infty$, asymptotic behaviour of the form $cR^2 + o(R^2)$, where $c$ is a positive constant.

Since $N^*(R) = N(R) + \sum_{n=n_0}^\infty N_0(R) + \sum_{n=n_0}^\infty N(R)$, we have

$$N^*(R) = (e + \varepsilon)(1 + \cdots),$$

whence

$$N^*(R) \leq (e + \varepsilon)(1 + \cdots).$$

Furthermore, for any $R > 0$,

$$N_0(R) \leq (e + \varepsilon)(1 + \cdots).$$

Since the length of a periodic saddle connection, $N(R) = \sum_{n=n_0}^\infty N(R) = \sum_{n=n_0}^\infty \sum_{i=1}^k \sum_{j=1}^{n_i} N_{\vec{v}_i}(R, |\vec{v}_i|/L_i)$, we have

$$\sum_{n=n_0}^\infty N(R) \leq (e + \varepsilon)(1 + \cdots).$$

So,

$$N^*(R) \leq (e + \varepsilon)(1 + \cdots).$$

Since $\varepsilon$ has been chosen arbitrarily,

$$N^*(R) \leq (e + \varepsilon)(1 + \cdots).$$

Combining the estimates,

$$N^*(R) \leq (e + \varepsilon)(1 + \cdots).$$

In a similar manner we can show that
\[ N^*(R) = N(R) + N(R/2) + \]

\[ \gamma^2, \]

\[ \text{clearly have } S(R) \leq S \cdot N(R), \]

\[ \text{on the distribution of periodic} \]

\[ \text{then each of the quantities} \]

\[ \text{asymptotic behaviour of} \]

\[ \text{any operators } a_1, \ldots, a_k \in \Gamma(\omega) \]

\[ \text{but } a_i \neq \pm 1, \text{ such that every} \]

\[ \text{h eigenvalues } \pm 1 \text{ is conjugate} \]

\[ \neq 0 \text{ (see [16]). Moreover, } \overline{v} \text{ is} \]

\[ \text{Without loss of generality we} \]

\[ \text{is uniquely determined. By} \]

\[ \text{ions (or periodic trajectories) by operators in } \Gamma(\omega), \text{ that is,} \]

\[ \text{tors collinear with these. Let} \]

\[ \text{parallel to } \overline{v}, \text{ let } L_1^1, \ldots, L_k^1 \text{ be} \]

\[ \text{to } \overline{v}, \text{ and let } S_1^1, \ldots, S_n^1 \]

\[ \text{affine automorphism of } \omega \text{ with} \]

\[ \text{between the saddle connections} \]

\[ \text{and } \gamma \overline{v}, \text{ respectively. Under} \]

\[ \text{ions and pencils are multiplied} \]

\[ \text{Thus, from what was} \]

\[ \sum_{1}^{k} \sum_{j=1}^{n_i} N^*_v(R \cdot |\overline{v}_i|/L_i), \]

\[ |/L_i), \]

\[ \gamma \overline{v}, \gamma \in \Gamma(\omega), \text{ of length} \]

\[ \text{is fixed under a non-identity} \]

\[ \text{tor of length } |\overline{v}|^{-1} \text{ orthogonal} \]

\[ \overline{v}/\overline{v} \] over all \( a \in \Gamma, a \neq \pm 1, \]

\[ \text{such that } a \overline{v} = \pm \overline{v}. \text{ Then} \]

\[ \frac{N^*_v(R)}{R^2} = (\alpha \cdot V(\Gamma))^{-1} \cdot R^2 + o(R^2) \text{ as } R \to \infty, \]

\[ \text{where } V(\Gamma) \text{ is the non-Euclidean area of a fundamental domain for the action of } \Gamma \]

\[ \text{on } \mathbb{H}^2. \]

\[ \text{This lemma, plus the formulae above it, give the required asymptotic behaviour} \]

\[ \text{for the quantities } N_0(R), N(R), S(R). \text{ Let } c \text{ be the constant in the asymptotic} \]

\[ \text{formula for } N(R). \text{ Then for any } \varepsilon > 0 \text{ and all values of } R \text{ exceeding a certain} \]

\[ R_0 = R_0(\varepsilon) \text{ the following inequality holds:} \]

\[ (c - \varepsilon) \cdot R^2 \leq N(R) \leq (c + \varepsilon) \cdot R^2. \]

\[ \text{Since } N^*(R) = N(R) + N(R/2) + \cdots, \text{ for any } n \in \mathbb{N} \text{ and } R \geq nR_0 \text{ we have} \]

\[ N^*(R) \geq (c - \varepsilon)(1 + 2^{-2} + \cdots + n^{-2}) \cdot R^2, \]

\[ \text{whence} \]

\[ \lim_{R \to \infty} \inf \frac{N^*(R)}{R^2} \geq c \cdot \sum_{n=1}^{\infty} n^{-2}. \]

\[ \text{Furthermore, for any } R \geq R_0 \text{ we have} \]

\[ N^*(R) \leq (c + \varepsilon)(1 + 2^{-2} + \cdots + n_0^{-2}) \cdot R^2 + \sum_{n > n_0} N(R/n), \quad n_0 = \lfloor R/R_0 \rfloor. \]

\[ \text{Since the length of a periodic trajectory is not smaller than the length of some} \]

\[ \text{saddle connection, } N(R) = 0 \text{ for } R < R_1 = m(\Gamma(\omega)). \text{ Hence,} \]

\[ \sum_{n > n_0} N(R/n) = \sum_{n_0 < n \leq \lfloor R/R_1 \rfloor} N(R/n) \leq N(R_0)R/R_1. \]

\[ \text{So,} \]

\[ N^*(R) \leq (c + \varepsilon) \sum_{n=1}^{\infty} n^{-2} \cdot R^2 + R \cdot N(R_0)/R_1. \]

\[ \text{Since } \varepsilon \text{ has been chosen arbitrarily,} \]

\[ \limsup_{R \to \infty} \frac{N^*(R)}{R^2} \leq c \cdot \sum_{n=1}^{\infty} n^{-2}. \]

\[ \text{Combining the estimates for } N^*(R)/R^2 \text{ we obtain} \]

\[ N^*(R) = c \cdot \frac{\pi^2}{6} \cdot R^2 + o(R^2) \text{ as } R \to \infty. \]

\[ \text{In a similar manner we can prove that} \]

\[ \lim_{R \to \infty} \frac{S^*(R)}{R^2} = \frac{\pi^2}{6} \cdot \lim_{R \to \infty} \frac{S(R)}{R^2}. \]


§4. Examples of Veech’s theorem

4.1. Planar structures on a torus.

Proposition 4.1. The stabilizer of a planar torus and of a planar torus with one singular point is a lattice.

Proof. As already noted in the proof of Theorem 3.4, the stabilizer of a planar torus does not change when a singular point is added to it. Furthermore, for arbitrary planar tori $\omega_1$ and $\omega_2$ on surfaces $M_1$ and $M_2$, respectively, we can find a map \( \varphi : M_1 \to M_2 \) which, in the local coordinates of the atlases of $\omega_1$ and $\omega_2$, is an affine map with linear part \( a \), det \( a > 0 \). Here, the stabilizers $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ are conjugate in $\text{SL}(2, \mathbb{R})$ : $\Gamma(\omega_2) = a \cdot \Gamma(\omega_1) \cdot a^{-1}$, so they are either both lattices or both not. Hence, it suffices to consider the planar torus $\mathbb{R}^2/\mathbb{Z}^2$. Clearly, $\Gamma(\mathbb{R}^2/\mathbb{Z}^2)$ contains the operator of rotation through the angle $\pi/2$. Moreover, since the horizontal trajectories form a single periodic pencil, of length equal to the width, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(\mathbb{R}^2/\mathbb{Z}^2)$, by Lemma 3.9. These two operators generate the modular group $\text{SL}(2, \mathbb{Z})$, which is a lattice in $\text{SL}(2, \mathbb{R})$ (see [16]). Since the stabilizer $\Gamma(\mathbb{R}^2/\mathbb{Z}^2)$ contains a lattice and is discrete (see Proposition 3.3), it is a lattice itself.

We note that for the planar torus $\mathbb{Z}_{r_1, r_2}$, the quantity $N^*(R)$ (see §3.4) is equal to half the number of non-zero vectors in $\mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2}$ of length at most $R$. Then the results of Bleher (see [17]) imply that

$$
N^*(R) = c \cdot R^2 + R^{1/2} \theta(R),
$$

where $c$ is a constant and $\theta(R)$ is a function that is almost-periodic in the sense of Besicovitch. This is a vast improvement of the estimate obtained in Theorem 3.13. It is not known whether such a representation exists for other planar structures with a lattice stabilizer.

4.2. Planar structures with one singular point. Let $n$ and $m$ be natural numbers, $2 \leq n \leq m$, $3 \leq m$. We put $k = \text{LCM}(n, m)$, $N = k/n$, $M = k/m$. Let $P$ and $Q$ be the regular $n$-gon, respectively $m$-gon, with equal sides, with one side of $P$ and one side of $Q$ horizontal. If $n = 2$ we consider only the regular $m$-gon; if $n$ and $m$ are odd, we require in addition that one of $P$ or $Q$ lies above its horizontal side and the other below it. Let $P_1, \ldots, P_N$ be the $n$-gons obtained from $P$ by rotation through the angles $0, \frac{2\pi}{N}, \ldots, (N - 1) \frac{2\pi}{N}$, and let $Q_1, \ldots, Q_M$ be the $m$-gons obtained from $Q$ by rotation through the angles $0, \frac{2\pi}{M}, \ldots, (M - 1) \frac{2\pi}{M}$. We consider the disjoint union $U$ of all $P_1, \ldots, P_N, Q_1, \ldots, Q_M$. In $U$ we identify the side $l_1$ of $P_i$ with the side $l_2$ of $Q_j$ if $l_1$ and $l_2$ are parallel and the polygons $P_i$ and $Q_j$ lie on different sides (for identical orientations on $l_1$ and $l_2$). If $n = 2$, we identify sides of $Q_1, \ldots, Q_M$ by this rule. By construction, each side of a polygon $P_1, \ldots, P_N, Q_1, \ldots, Q_M$ is identified with precisely one other side, and after the identification the union $U$ becomes a connected, oriented, compact surface $M_{n,m}$. The identity maps on the interiors of the polygons $P_1, \ldots, P_N, Q_1, \ldots, Q_M$ can be uniquely extended to a planar structure $\omega_{n,m}$ on $M_{n,m}$. All vertices of the polygons are identified, giving one point in $M_{n,m}$. This point is the unique singular point of $\omega_{n,m}$. It is removed from the sides of the polygons $P_i$ and $Q_j$ at connections of $\omega_{n,m}$.

Proposition 4.2. The polygons $P_i$ and $Q_j$, for $1 \leq i \leq N$ and $1 \leq j \leq M$, are $n$- and $m$-gons making up regular polygons in the regular $m$-gon. We may assume that the regular polygon to the right of the centres of adjoining polygons $P_i$ and $Q_j$, these line segments are saddle connections of $\omega_{n,m}$. $\pi = \frac{\pi}{n}, \pi = \frac{\pi}{m}$, and $\pi = \frac{\pi}{m}$, $\pi = \frac{\pi}{m}$. By considering the side lie symmetrically on the line segments (which is equal to $2n \cdot m$), generated by the linear element $\Gamma(n) \in \Gamma_{n,m}$ in the decomposition of $\omega_{n,m}$.

We let $T$ be such that in the decomposition to the linear part $R$. We may assume that the polygons $Q_1, \ldots, Q_M$ are determined by the element $R$ onto the set of triangles $\omega_{n,m}$. Then $r_1 \circ r_2^{-1}$ is the linear construction in §2.2 implies $\omega_{n,m} = \Gamma_{n,m}^+$. We will now give some results.

Let $n, m$ be integers, $n, m \geq 2$.

$$
L_{n,m} = \left( \begin{array}{cc}
(n) & \sigma_n \\
\sigma_n & (m)
\end{array} \right)
$$

We denote by $\Gamma_{n,m}$ (or $\Gamma_{n,m}^+$).

$$
\sigma_n = \left( \begin{array}{cc}
(n) & \sigma_n \\
\sigma_n & (m)
\end{array} \right)
$$

We denote by $\Gamma_{n,m}$ (or $\Gamma_{n,m}^+$) for $n$ even.

Proposition 4.3. The group $\Gamma_{n,m}$ is a lattice in $\text{SL}(2, \mathbb{R})$.

Proof. We consider the group $\Gamma_{n,m}$ in the group $\text{SL}(2, \mathbb{R})$ acting
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Theorem

and of a planar torus with one

3.4, the stabilizer of a planar

3. jaded to it. Furthermore, for

$M_2$, respectively, we can find

$s$ of the atlases of $\omega_1$ and $\omega_2$, e.g., the stabilizers $\Gamma(\omega_1)$ and $\cdot a^{-1}$, so they are either both

e planar torus $\mathbb{R}^2/\mathbb{Z}^2$. Clearly, the

angle $\pi/2$. Moreover, since a

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two operators generate the

(see [16]). Since the stabilizer

position 3.3), it is a lattice itself.

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length at most $R$. Then the

$R$.

almost-periodic in the sense of

of the obtained in Theorem 3.13.

is for other planar structures.

int. Let $n$ and $m$ be natural

$n,m)$. $N = k/n, M = k/m,$

$m$, with equal sides, with one

we consider only the regular

that one of $P$ or $Q$ lies above

$\ldots, P_N$ be the $n$-gons obtained

$-1)^{\frac{1}{2k}}$, and let $Q_1, \ldots, Q_M$ be

angles $0, \frac{2\pi}{k}, \ldots, (M - 1) \frac{2\pi}{k},$

$Q_1, \ldots, Q_M$. In $U$ we identify

e parallel and the polygons $P_1$

$\ldots, P_N$ on $l_1$ and $l_2$. If $n = 2$, we

action, each side of a polygon

one other side, and after the

en, compact surface $M_{n,m}$.

$P_1, \ldots, P_N, Q_1, \ldots, Q_M$ can be

$i$. All vertices of the polygons

of $\omega_{n,m}$. It is removable if $n = m = 3$ or if $n = 2$ and $m = 3, 4$ or $6$. The

sides of the polygons $P_1, \ldots, P_N, Q_1, \ldots, Q_M$, and only they, are shortest saddle

connections of $\omega_{n,m}$.

Proposition 4.2. The planar structure corresponding to the triangle with angles

$\frac{\pi}{n}, \frac{\pi}{m}, \pi - \frac{\pi}{n} - \frac{\pi}{m}$ ($n, m \in \mathbb{N}, n \leq m$) coincides with $\omega_{n,m}$ up to rotation and removal of certain removable singular points.

Proof. We add to $\omega_{n,m}$ as removable singular points the centres of the regular

$n$- and $m$-gons making up the surface $M_{n,m}$ (for $n = 2$, the regular 2-gon is a side of

the regular $m$-gon). We obtain a planar structure $\overline{\omega}_{n,m}$. We join the centre of each

regular polygon to the vertices of this polygon by line segments, and also join the

centres of adjoining polygons by the mid-perpendicular to their common side. All

these line segments are disjoint (except perhaps for common end-points). They are

saddle connections of $\overline{\omega}_{n,m}$ and partition $M_{n,m}$ into equal triangles with angles $\frac{\pi}{n},$

$\frac{\pi}{m}, \pi - \frac{\pi}{n} - \frac{\pi}{m}$. By construction, triangles in the decomposition having a common

side lie symmetrically with respect to this side, and the number of such triangles

(which is equal to $2n \cdot N = 2m \cdot M = 2k$) is equal to the order of the group $R$

generated by the linear parts of the reflections with respect to the sides of any

triangle in the decomposition.

We let $T$ be one such triangle. With each $r \in R$ we can associate the triangle $T_r$

in the decomposition to which $T$ is mapped by the affine transformation $f_r$ with

linear part $r$. We may assume that the sets of centres of the polygons $P_1, \ldots, P_N$

and of the polygons $Q_1, \ldots, Q_M$ are fixed under $f_r$. Thus, the triangle $T_r$ is uniquely

determined by the element $r$, and the correspondence $r \mapsto T_r$ is a bijection from

$R$ onto the set of triangles in the partition; if $T_{r_1}$ and $T_{r_2}$ are adjoining triangles,

then $r_1 \circ r_2^{-1}$ is the linear part of the reflection in their common side. Then the

construction in §2.2 implies that $\overline{\omega}_{n,m}$ is the planar structure corresponding to $T$.

We will now give some examples of subgroups of $\text{SL}(2, \mathbb{R})$ that are lattices. Let

$n, m$ be integers, $n, m \geq 2$, $(n,m) \neq (2,2)$. We put

$$L_{n,m} = \frac{\cos \frac{\pi}{n} + \cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}, \quad L_{n,\infty} = \frac{\cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{n}} = \cot \frac{\pi}{2n}.$$ We denote by $\Gamma_{n,m}$ (or $\Gamma^+_{n,m}$) the subgroup of $\text{SL}(2, \mathbb{R})$ generated by the elements

$$\sigma_n = \left( \begin{array}{cc} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{array} \right) \quad \text{and} \quad \tau_{n,m} = \left( \begin{array}{cc} 1 & 2L_{n,m} \\ 0 & 1 \end{array} \right).$$ We denote by $\Gamma^-_{n,m}$ the subgroup generated by the elements $-\sigma_n$ and $\tau_{n,m}$

($\Gamma_{n,m} = \Gamma^+_{n,m}$ for $n$ even).

Proposition 4.3. The groups $\Gamma^\pm_{n,m}$ ($2 \leq n < \infty, 2 \leq m \leq \infty, (n,m) \neq (2,2)$) are lattices in $\text{SL}(2, \mathbb{R})$.

Proof. We consider the Lobachevskii plane $\mathbb{H}^2 = \{ z \in \mathbb{C} | \text{Im} \, z > 0 \}$, on which the group $\text{SL}(2, \mathbb{R})$ acts (see the proof of Lemma 3.7). Through the point $i$ we
draw two non-Euclidean straight lines $l_1$ and $l_2$ making angles $\frac{\pi}{n}$ with the geodesic joining $i$ and $\infty$ ($l_1$ and $l_2$ are Euclidean circles with centres at the points $\cot \frac{\pi}{n}$ and $-\cot \frac{\pi}{n}$, respectively; $l_1 = l_2$ if $n = 2$). Let $A_1$ be the point of intersection of $l_1$ and the line $Re z = L_{n,m}$, and let $A_2$ be the point of intersection of $l_2$ and the line $Re z = -L_{n,m}$. The imaginary parts of $A_1$ and $A_2$ are equal to $\sin \frac{\pi}{m}$ for $m < \infty$, while for $m = \infty$ both $A_1$ and $A_2$ belong to the absolute $Im z = 0$. The non-Euclidean 4-gon $W$ with vertices $i$, $A_1$, $\infty$, $A_2$ has finite non-Euclidean area (see [16]). The operator $\sigma_n$ (or $-\sigma_n$) maps the side $iA_2$ of $W$ to the side $iA_1$, leaving $i$ fixed. The operator $\tau_{n,m}$ maps the side $A_2\infty$ to the side $A_1\infty$, leaving $\infty$ fixed. The angle of $W$ at $i$ is $\frac{2\pi}{n}$, those at $A_1$ and $A_2$ are $\frac{\pi}{m}$ for $m < \infty$ and $0$ for $m = \infty$. In view of the above, a theorem of Poincaré (see [16]) implies that each group $\Gamma_{n,m}^\pm$ is discrete and that $W$ is a fundamental domain for its action on $\mathbb{H}^2$.

Since $W$ has finite area, $\Gamma_{n,m}^\pm$ is a lattice.

We note that the modular group $SL(2, \mathbb{Z})$, used in the proof of Proposition 4.1, is among the listed examples of lattices; it is the group $\Gamma_{2,3}$.

![Figure 1. $\omega_{2,5} = \omega_{5,5}$](image)

**Theorem 4.4.** The stabilizers $\Gamma(\omega_{2,n})$ and $\Gamma(\omega_{n,n})$ contain the group $\Gamma_{n,2}$ for $n$ odd and the group $\Gamma_{n/2,\infty}$ for $n$ even. For an arbitrary $n$ we have $\Gamma_{n,3} \subset \Gamma(\omega_{n,2n})$. Moreover, $\Gamma_{6,\infty} \subset \Gamma(\omega_{3,4})$, $\Gamma_{15,\infty} \subset \Gamma(\omega_{3,5})$. All stabilizers listed here are lattices.

**Proof.** After rotation through an angle $\frac{2\pi}{n}$ or $\frac{2\pi}{m}$, each of the regular $n$- and $m$-gons making up $M_{n,m}$ can be made to coincide by a shift with one of the polygons listed. By the construction of $M_{n,m}$, this transformation is well defined on $M_{n,m}$, and is an affine automorphism of the planar structures $\omega_{n,m}$ and $\omega_{n,m}$. Thus, the operators of rotation through $\frac{2\pi}{n}$ and $\frac{2\pi}{m}$ belong to the stabilizers $\Gamma(\omega_{n,m})$ and $\Gamma(\omega_{n,m})$. In particular, we find that $\Gamma(\omega_{2,n})$ contains $\sigma_n$ for $n$ odd; $\Gamma(\omega_{2,n})$ and $\Gamma(\omega_{n,n})$ contain $\sigma_{n/2}$ for $n$ even; $\Gamma(\omega_{2,n})$ contains $\sigma_n$; and $\sigma_6 \in \Gamma(\omega_{3,4}), -\sigma_6 \in \Gamma(\omega_{3,5})$.

We now turn to the horizontal trajectories of $\omega_n$ here and below, identifying parallel segments to the width of each pencil. Let $\Gamma_{n,2} \subset \Gamma(\omega_{n,n})$ for $n$ odd. For $n$ odd, $\omega_{2,n}$ coincides with a periodic pencil. If $n$ is even, the width of each pencil is $2\cot \frac{\pi}{n}$ (see Fig. 2). For this pencil with $M_{2,n}$, which $M_{2,n}$ is made up of $\Gamma(\omega_{2,n})$ contains $\tau_{n/2,\infty}$ of the trajectories of $\omega_{2,n}$.
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The proof of Proposition 4.1, up $\Gamma_{2,3}$.

contain the group $\Gamma_{n,2}$ for $n$ even and $\omega_n$ we have $\Gamma_{n,3} \subseteq \Gamma(\omega_{n,2n})$. The lattices listed here are lattices.

We now turn to the horizontal trajectories of these planar structures. The horizontal trajectories of $\omega_{n,n}$ split into $\lceil n/2 \rceil$ pencils of periodic trajectories (see Fig. 1; here and below, identical digits indicate identified sides). The ratio of the length to the width of each pencil is $2 \cot \frac{\pi}{n}$, that is, by Lemma 3.9, $\tau_{2,2} \in \Gamma(\omega_{n,n})$. Then $\Gamma_{n,2} \subseteq \Gamma(\omega_{n,n})$ for $n$ odd. For $n$ even we have $\tau_{n,2} = \tau_{n/2,\infty}$, and $\Gamma_{n/2,\infty} \subseteq \Gamma(\omega_{n,n})$. For $n$ odd, $\omega_{2,n}$ coincides with $\omega_{n,n}$; for $n$ even its horizontal trajectories form $[n/4]$ periodic pencils. If $n$ is not divisible by 4, then the ratio of the length to the width of each pencil is $2 \cot \frac{\pi}{n}$. If $n$ is divisible by 4, all pencils except one have this ratio (see Fig. 2). For this pencil, which contains the centre of the regular $n$-gon (from which $M_{2,n}$ is made up), the ratio of the length to the width is $\cot \frac{\pi}{n}$. In any case, $\Gamma(\omega_{2,n})$ contains $\tau_{n/2,\infty}$, and so $\Gamma_{n/2,\infty} \subseteq \Gamma(\omega_{2,n})$. Furthermore, the horizontal trajectories of $\omega_{n,3n}$ form $n - 1$ periodic pencils (see Fig. 3), the ratio of the length...
to the width for each of these being

\[ \cot \frac{\pi}{2n} + \cot \frac{\pi}{n} = \frac{\cot \frac{\pi}{2n} \sin \frac{\pi}{n} + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} = \frac{(1 + \cos \frac{\pi}{n}) + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} = 2L_{n,3}. \]

By Lemma 3.9 we have \( \tau_{n,3} \in \Gamma(\omega_{n,2n}) \), and \( \Gamma(\omega_{n,2n}) \supset \Gamma_{n,3} \).

The horizontal trajectories of \( \omega_{3,4} \) split into 3 periodic pencils (see Fig. 4). The two pencils bounded by shortest saddle connections (hatched in the figure) have length \( 2d(1 + \cos \frac{\pi}{6}) \) and width \( d \sin \frac{\pi}{6} \), where \( d \) is the length of the shortest saddle connection. So, the ratio of length to width is equal to \( 2L_{6,\infty} \). The third pencil is readily seen to have this ratio equal to \( 2 \cot \frac{\pi}{12} \). Consequently, \( \tau_{6,\infty} \in \Gamma(\omega_{3,4}) \) and \( \Gamma(\omega_{3,4}) \supset \Gamma_{6,\infty} \).

The horizontal trajectories of \( \omega_{3,5} \) split into 4 periodic pencils (see Fig. 5). The length and width of the pencil indicated by straight hatching in the figure are equal to \( 2d(1 + \cos \frac{\pi}{15}) \) and \( d \sin \frac{\pi}{15} \), where \( d \) is the length of the shortest saddle connection. The length and width of the pencil indicated by skew hatching in the figure are equal to \( 2D(1 + \cos \frac{\pi}{15}) \) and \( D \sin \frac{\pi}{15} \), where \( D \) is the length of the diagonal of the regular 5-gon with side \( d \). In both cases the ratio of length to width is equal to \( 2L_{15,\infty} \). The ratio of length to width for the pencil indicated by the broken lines in the figure is equal to \( 2 \cot \frac{\pi}{30} = 2L_{15,\infty} \). Finally, the pencil not marked at all has length \( 2d(1 + 2 \cos \frac{\pi}{15} + \cos \frac{2\pi}{15}) \) and width \( d \sin(\pi/15) \). Their ratio is

\[ \frac{2(1 + 2 \cos \frac{\pi}{15} + \cos \frac{2\pi}{15})}{\sin \frac{2\pi}{15}} = \frac{2(2 \cos^2 \frac{\pi}{15} + 2 \cos \frac{\pi}{15})}{2 \sin \frac{\pi}{15} \cos \frac{\pi}{15}} = 2 \frac{\cos \frac{\pi}{15} + 1}{\sin \frac{\pi}{15}} = 2L_{15,\infty}. \]

So, by Lemma 3.9, \( \tau_{15,\infty} \) for each of these planar structures. The stabilizers are discrete.

We conclude by noting that the planar structures conjectured for \( \omega_{n,2n} \) (n = 7, 11, 13) are possible to find examples.

**Theorem 4.5. The stable \( \omega_{3,4} \) and \( \omega_{3,5} \) structures.**

**Proof.** The horizontal trajectories (see Fig. 6). The ratio of length to width for the pencil not marked in the hatching in the figure is \( d \sqrt{2} \sin \frac{\pi}{12} / L_{15,\infty} = d(\sqrt{3} - 1) / L_{15,\infty} \). The shortest saddle connections are incommensurate, hence the pencil is not a lattice.
\[
\frac{\cos \frac{\pi}{n} + \cos \frac{\pi}{m}}{\sin \frac{\pi}{n}} = 2L_{n,3},
\]

\[\varphi(n) \supset \Gamma_{n,3}.\]

Figure 5. \(\omega_{3,5}\), rotated through 90°

So, by Lemma 3.9, \(\tau_{15,\infty} \in \Gamma(\omega_{3,5})\) and \(\Gamma(\omega_{3,5}) \supset \Gamma_{15,\infty}\). Thus, the stabilizer of each of these planar structures contains a lattice. Hence, in view of the fact that the stabilizers are discrete, they themselves must be lattices.

We conclude by noting that in [1] the presence of a lattice stabilizer was proved for the planar structures \(\omega_{2,n}\) and \(\omega_{n,n}\) (\(n \geq 3\)), was announced for \(\omega_{3,4}\), and was conjectured for \(\omega_{n,2n}\) (\(n \geq 2\)).

4.3. Non-lattice stabilizers. Theorem 3.4 (together with Lemma 3.8) makes it possible to find examples of planar structures whose stabilizers are not lattices.

Theorem 4.5. The stabilizers of \(\omega_{4,6}\) and \(\omega_{4,12}\) are not lattices in \(\text{SL}(2, \mathbb{R})\).

Proof. The horizontal trajectories of \(\omega_{4,6}\) split into 4 pencils of periodic trajectories (see Fig. 6). The ratio of the length to the width for the pencil indicated by skew hatching in the figure is equal to \(r_1 = 1 + \cot \frac{\pi}{6} = 1 + \sqrt{3}\). On the other hand, the pencil not marked in the figure has length \(2d(1 + \sqrt{2} \cos \frac{\pi}{12}) = d(3 + \sqrt{3})\) and width \(d\sqrt{2}\sin \frac{\pi}{12} = d(\sqrt{3} - 1)/2\) (as in the proof of Theorem 4.4, \(d\) is the length of the shortest saddle connection). Their ratio is \(r_2 = 6 + 4\sqrt{3}\). The numbers \(r_1\) and \(r_2\) are incommensurate, hence, by Theorem 3.4 and Lemma 3.8, the stabilizer \(\Gamma(\omega_{4,6})\) is not a lattice.
5.1. Finiteness of the number of covers

Definition 5.1. Let \( \omega \) be a cover of \( M \). \( \omega_1 \) is said to cover \( \omega_2 \) if there is a mapping \( \alpha_i \) of \( M_1 \) into \( M_2 \) in the local coordinates.

In view of the compactness of the preimages of a point \( x \) on the surfaces, this number is called the \textit{multiplicity} of the cover, \( m \). If \( x_1, \ldots, x_k \in M_1 \) are its preimages, then \( m_1 \) divides each \( m_1 \) and \( m_2 \).

The following construction helps us to partition a rational polygon \( \omega \). The vertices of these intersecting only along a whole \( \frac{m}{2} \) -gons with a common side lying in the plane of \( \omega \) and assign to each \( \alpha_i \) a + or - sign in accordance with the order of \( \alpha_i \) and \( \alpha_0 \) in the previous ones. The condition of the multiplicities \( m_0 \) and \( m_1 \) on certain subpolygons.

Proposition 5.1. The conditions of the partitioning \( \omega \) to \( \omega_0 \) or removal from \( \omega_0 \) depend only on the multiplicities of the covers.

Proof. From the construction, we see that \( \alpha_i \) maps \( P_i \) to \( P_i \) and has the symmetry with respect to \( \omega \). Consider a single continuous map \( \alpha \).

Let \( R(Q), R(P_1), \ldots \) be the reflections in the sides of \( P \). By construction, we have that \( R(Q) \subset R(P) \).

Write \( f(x, t, \alpha_1^{-1}) \) where \( i \) is a reflection in the \( \alpha_i \). Then the multiplicity \( m_Q \) and \( m_P \) and the number \( f(x, t) \) is the index of the subgroup \( \Gamma(Q) \subset R(P) \). To correct this situation, divide \( \omega \) by \( \rho \).

Definition 5.2. Let \( \omega \) be a cover of \( \omega \). Two covers of \( \omega \) are said to be \textit{incommensurate} if they cannot be obtained from each other by translating and rotating. Two covers of \( \omega \) are said to be \textit{commensurate} if they can be obtained from each other by translating and rotating.

One can similarly treat the case \( \omega_{4, 12} \). Its horizontal trajectories split into 4 periodic pencils (see Fig. 7). The ratio of the length to the width for the pencil marked by straight hatching in the figure is equal to \( r_1 = 1 + \cot \frac{\pi}{12} = 3 + \sqrt{3} \). The pencil marked by skew hatching in the figure has length \( 2d(1 + 2 \cos \frac{\pi}{6} + \cos \frac{\pi}{3}) \) and width \( d \sin \frac{\pi}{6} \); their ratio is \( r_2 = 6 + 4\sqrt{3} \). The numbers \( r_1 \) and \( r_2 \) are incommensurate again, hence the stabilizer \( \Gamma(\omega_{4, 12}) \) is not a lattice.
§5. Covers of planar structures

5.1. Finiteness of the number of covers.

Definition 5.1. Let \((M_1, \omega_1)\) and \((M_2, \omega_2)\) be surfaces with planar structures. \(\omega_1\) is said to cover \(\omega_2\) if there is a continuous map \(f: M_1 \to M_2\) (called a cover) mapping the singular points of \(\omega_1\) to the singular points of \(\omega_2\) and acting as a shift in the local coordinates given by the atlases of \(\omega_1\) and \(\omega_2\).

In view of the compactness of the surfaces under consideration, the number of preimages of a point \(x \in M_2\) under \(f\) is finite. In view of the connectedness of the surfaces, this number is the same for all non-singular points. It is called the multiplicity of the cover \(f\). If \(x \in M_2\) is a singular point of multiplicity \(m\) and \(x_1, \ldots, x_k \in M_1\) are its preimages with respective multiplicities \(m_1, \ldots, m_k\), then \(m\) divides each \(m_i\) and \(\sum_{i=1}^{k} m_i = mN\), where \(N\) is the multiplicity of the cover.

The following construction gives examples of covers of planar structures. Suppose we partition a rational polygon \(Q\) into equal polygons \(P = P_1, P_2, \ldots, P_n\), any two of these intersecting only along common sides and at common vertices, and polygons with a common side lying symmetrically with respect to this side. Moreover, we assign to each \(P_i\) a + or − sign in such a way that polygons with a common side have different signs (if \(P\) is not axially symmetric, this condition follows from the previous ones). The construction in §2.2 associates with \(P\) and \(Q\) planar structures \(\omega_P\) and \(\omega_Q\) on certain surfaces \(M_P\) and \(M_Q\).

Proposition 5.1. The planar structure \(\omega_Q\) covers \(\omega_P\), possibly after the addition to \(\omega_Q\) or removal from \(\omega_P\) of a certain number of removable singular points. The multiplicity of the cover is a divisor of \(n\).

Proof. From the construction it follows that there are affine maps \(\alpha_1, \ldots, \alpha_n\) such that \(\alpha_i\) maps \(P_i\) to \(P\) and, for arbitrary \(P_i\) and \(P_j\) with a common side, \(\alpha_j^{-1} \alpha_i\) is the symmetry with respect to this side. The maps \(\alpha_1, \ldots, \alpha_n\) can be extended to a single continuous map \(f: Q \to P\).

Let \(R(Q), R(P_1), \ldots, R(P_n)\) be the groups generated by the linear parts of the reflections in the sides of the polygons. By construction, \(R(P_1) = \cdots = R(P_n)\), so that \(R(Q) \subset R(P)\). We define the map \(g: Q \times R(Q) \to P \times R(P)\) by \(g(x, r) = (f(x), r\alpha_i^{-1})\), where \(i\) is such that \(x \in P_i\) and \(\alpha_i\) is the linear part of \(\alpha_i\) (clearly, \(\alpha_i \in R(P)\)). Then the spaces \(Q \times R(Q)\) and \(P \times R(P)\) can be factored to surfaces \(M_Q\) and \(M_P\) and the map \(g\) can be factored to a well-defined map \(h: M_Q \to M_P\). Now, \(h\) is a cover of the planar structures \(\omega_Q\) and \(\omega_P\) of multiplicity \(n/m\), where \(m\) is the index of the subgroup \(R(Q)\) in \(R(P)\), except that certain non-singular points on \(M_Q\) can be mapped to (removable) singular points on \(M_P\). To be precise, these are points of the form \((x, r)\), where \(x\) is a vertex of a polygon \(P_i\) that is not a vertex of \(Q\). To correct this situation it suffices to add such points to the singular points of \(\omega_Q\). If none of the points in the preimage \(f^{-1}(f(x))\) is a vertex of \(Q\), instead of this we may regard the points \((f(x), r) \in M_P\) as singular for \(\omega_Q\).

Definition 5.2. Let \((M, \omega), (M_1, \omega_1)\) and \((M_2, \omega_2)\) be surfaces with planar structures. Two covers of planar structures \(f_1: M_1 \to M\) and \(f_2: M_2 \to M\) are called
Proposition 5.2. A planar structure $\omega$ on a surface $M$ admits only a finite number of covers of given multiplicity, up to isomorphism. This number is bounded by a constant depending on the genus of $M$, the number of removable singularities of $\omega$, and the multiplicity of the cover.

Proof. We consider an arbitrary triangulation of $M$ for which the vertices are the singular points of $\omega$, the edges are the saddle connections, and the faces do not contain interior singular points. The proof of Theorem 2.9 implies that the number of edges of the triangulation is $3(k - \chi)$, where $k$ is the number of singular points of $\omega$ and $\chi$ is the Euler characteristic of $M$. We choose edges (saddle connections) $L_1, \ldots, L_n$ such that the domain $M_0$ obtained from $M$ by deleting the singular points and the selected saddle connections becomes connected and simply-connected. We cut $M$ along each $L_i$, and we denote the boundaries of the cuts by $L_i^+$ and $L_i^-$. By adding to $M_0$ the boundaries of the cuts we obtain a compact simply-connected surface $M'$ with boundary. We now take $N$ copies of $M'$ (that is, the direct product $M' \times \{1, 2, \ldots, N\}$), given by permutations $\pi_1, \ldots, \pi_n$ on $\{1, 2, \ldots, N\}$, and identify the segments $L_i^+ \times \{j\}$ and $L_i^- \times \{i\}$, $1 \leq i \leq n$, $1 \leq j \leq N$. We obtain a compact surface $M'' = M''(\pi_1, \ldots, \pi_n)$ without boundary. The surface $M''$ is connected if the group generated by the permutations $\pi_1, \ldots, \pi_n$ acts transitively on $\{1, 2, \ldots, N\}$. The planar structure on $M_0 \times \{1, 2, \ldots, N\} \subseteq M''$ induced by restricting $\omega$ to $M_0$ can be uniquely extended to a planar structure $\omega(\pi_1, \ldots, \pi_n)$ on $M''$. Moreover, the natural projection $p: M'' \to M$ is an $N$-fold cover of the planar structures $\omega(\pi_1, \ldots, \pi_n)$ and $\omega$. The total number of covers thus constructed does not exceed $(N!)^n$. By construction, $n \leq 3(k - \chi)$. By Theorem 2.9, the number of non-removable singular points of $\omega$ is at most $-\chi$. Hence, $(N!)^n$ is bounded by a constant depending on $\chi$, $N$ and the number of removable singular points of $\omega$.

To finish the proof it now suffices to show that every $N$-fold cover $f$ of $\omega$ by some planar structure $\omega_1$ on a surface $M_1$ is isomorphic to one of the covers constructed above. We take a point $x \in M_0$ and let $x_1, \ldots, x_N$ be its preimages under $f$. Since $M_0$ is simply-connected, there are unique maps $f_1, \ldots, f_N$ from $M_0$ into $M_1$ that are right-inverse to $f$ and such that $f_i(x) = x_i$, $1 \leq i \leq N$. Moreover, the domains $f_1(M_0), \ldots, f_N(M_0)$ are pairwise disjoint and are separated from each other by the preimages of the saddle connections $L_1, \ldots, L_n$ (each saddle connection has $N$ preimages). This implies that the map $\varphi: M_0 \times \{1, 2, \ldots, N\} \to M_1$ given by $\varphi(x, i) = f_i(x)$ can, for a certain choice of permutations $\pi_1, \ldots, \pi_n$, be extended to a continuous map $\varphi: M''(\pi_1, \ldots, \pi_n) \to M_1$. By construction, $f \circ \varphi = p$ and $\varphi$ is an isomorphism of the planar structures $\omega(\pi_1, \ldots, \pi_n)$ and $\omega_1$. Thus, the cover $f$ is isomorphic to the cover $p$.

Proposition 5.3. The number of covers realized by a planar structure $\omega$ is finite, up to isomorphism, and bounded by a constant depending on the genus of $M$ and the number of singular points of $\omega$.

Proof. Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be the sets of points $x \in M_1$ and $y \in M_2$ such that $f_1 = f_2 \circ \varphi$. We can similarly define the isomorphism of covers $g_1: M \to M_1$ and $g_2: M \to M_2$.

Theorem 5.4. Suppose $\omega$ is a planar structure. Then the group of automorphisms $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ are both not.

Proof. Let $M_1$ be the group $\Gamma(M, \omega, f)$ with $\omega$ a planar structure $\omega$ by $\omega_1$, and let $\varphi: M_0 \to M_1$ be a cover. The map $f$ real $\varphi: M_0 \to M_2$, $a \in \text{GL}(2, \mathbb{R})$. Furthermore, the automorphisms of $\omega_1$ are given by $a(\omega)$ is the line $S: (M, \omega, f) \varphi = (M, a(\omega))$ isomorphic to $\omega_1$. Moreover, Proposition 5.2 the group $\Gamma(\omega)$ acts as the identity automorphisms in $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ of finite index in $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$. It follows that $\Gamma(\omega_1)$ and $\Gamma(\omega_2)$ are both not.
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y $\omega_2$, and let $\tilde{S}$ be the set of such triples regarded up to isomorphism of covers. The map $f$ realizes also a cover of the planar structures $\omega_1$ and $\omega_2$, with $a \in \text{GL}(2, \mathbb{R})$. Furthermore, an arbitrary element $\varphi$ of the group $G$ of affine au-

omorphisms of $\omega_1$ is an isomorphism of the planar structures $\omega_1$ and $a(\varphi)^{-1}\omega_1$, where $a(\varphi)$ is the linear part of $\varphi$. Thus, a right action of $G$ is defined on $S : (M, \omega, f, \varphi) = (M, a(\varphi)^{-1}\omega, f \circ \varphi)$. We note that isomorphic covers are mapped to isomorphic covers, that is, the action factorizes to an action of $G$ on $\tilde{S}$. Since by Proposition 5.2 the set $\tilde{S}$ is finite, there is a subgroup $G_0 \subset G$ of finite index that acts as the identity on $\tilde{S}$. Let $\Gamma$ be the group of linear parts of the affine au-

omorphisms in $G_0$ that preserve the orientation on $M_1$. Clearly, $\Gamma$ is a subgroup of finite index in $\Gamma(\omega_1)$. On the other hand, since $G_0$ acts as the identity on $\tilde{S}$, it follows that $\Gamma$ is contained in the stabilizer of every planar structure covered by $\omega_1$. In particular, $\Gamma \subset \Gamma(\omega_1) \cap \Gamma(\omega_2)$, that is, $\Gamma(\omega_1) \cap \Gamma(\omega_2)$ is a subgroup of finite index in $\Gamma(\omega_1)$.
The finiteness of the index of $\Gamma(\omega_1) \cap \Gamma(\omega_2)$ in $\Gamma(\omega_2)$ can be similarly derived from Proposition 5.3.

Proposition 5.5. The stabilizer of a planar structure corresponding to a regular $n$-gon is a lattice.

Proof. By joining the centre of the regular $n$-gon to the vertices and the mid-points of the sides using straight line segments, we obtain a decomposition of the regular $n$-gon into triangles with angles $\pi/2$ and $\pi/n$. Proposition 5.1 implies that the planar structure $\omega_n$ corresponding to the regular $n$-gon covers the planar structure $\tilde{\omega}_n$ corresponding to a triangle in the decomposition; moreover, in the latter we have to take into account the non-singular points arising from the vertices with angles $\pi/2$ and $\pi/n$. With this understanding, $\Gamma(\tilde{\omega}_n)$ is a lattice, as has been shown in §4.2. By Theorem 5.4, $\Gamma(\omega_n)$ is also a lattice.

Proposition 5.5 was proved by Veech in [2], where he also studied the planar structures $\omega_n$ and their stabilizers in some detail.

The following example shows that we have to consider removable singularities. An isosceles triangle with angle $2\pi/n$ at the vertex is partitioned by the altitude into two triangles with angles $\pi/2$ and $\pi/n$. Therefore the planar structure $\omega_n'$ corresponding to it covers $\tilde{\omega}_n$, where in this case we have to take as non-singular the points corresponding to the angle $\pi/2$ but not those corresponding to $\pi/n$. We can show that with this understanding, $\Gamma(\omega_n')$ is not a lattice for any odd $n > 3$; consequently, $\Gamma(\omega_n')$ is also not a lattice. In particular, it is not known whether $\omega_n'$ is an elementary planar structure for these values of $n$.

We give a sufficient condition for $\Gamma(\omega) \subset \Gamma(\tilde{\omega})$, given that $\omega$ covers $\tilde{\omega}$.

Proposition 5.6. Let $f$ be a cover of a planar structure $\tilde{\omega}$ by a planar structure $\omega$, having multiplicity $N$. If all covers of multiplicity $N$ realized by $\omega$ are pairwise isomorphic, then $\Gamma(\omega) \subset \Gamma(\tilde{\omega})$. In particular, this is true if for some $\overline{v} \in \mathbb{R}^2$ the planar structure $\tilde{\omega}$ has a unique saddle connection with development $\overline{v}$; for example, $\Gamma(\omega_n) \subset \Gamma(\tilde{\omega}_n)$.

Proof. We assume that all covers of multiplicity $N$ realized by $\omega$ are isomorphic to $f$. We choose an element $a \in \Gamma(\omega)$ and let $\varphi$ be an affine automorphism of $\omega$ with linear part $a^{-1}$. Then $f \circ \varphi$ is a cover of the planar structures $\omega$ and $\tilde{\omega}$. The multiplicity of $f \circ \varphi$ is $N$, hence $f$ and $f \circ \varphi$ are isomorphic. This implies that $\tilde{\omega}$ and $a \tilde{\omega}$ are isomorphic, that is, $a \in \Gamma(\tilde{\omega})$. Since $a$ is arbitrary, this implies that $\Gamma(\omega) \subset \Gamma(\tilde{\omega})$.

Now we assume that $\tilde{\omega}$ has a unique saddle connection with development $\overline{v}$. Then $\omega$ has exactly $N$ saddle connections with this development. Clearly, for any cover of multiplicity $N$ these saddle connections become a single saddle connection. Therefore, the proof of Proposition 5.3 implies that all such covers are isomorphic.

We finally show that $\Gamma(\omega_n) \subset \Gamma(\tilde{\omega}_n)$. In fact, $\omega_n$ covers $\tilde{\omega}_n$, and all shortest saddle connections of $\tilde{\omega}_n$ (their preimages under the cover correspond to the sides of the regular $n$-gon) have distinct developments.

§6. A property of $\omega$.

6.1. Property A.

Definition 6.1. Let $\omega$ be a $\omega$-triangle on $M$ whose vertices and whose interior do not contain.

Definition 6.2. A planar structure $\omega$ is said to have a saddle connection if either periodic or saddle connection is a saddle connection.

Proposition 6.1. Property $A$ has a saddle connection in $\omega$.

Proof. We assume that $\omega$ is larger than the area of every $\omega$-triangle. Through a point $x \in L$ with $l$ is the length of $L$, perpendicular to $L$ not equal to $1/2 : l \cdot \varepsilon = \delta$.

There is an arbitrary saddle connection in $\omega$. On the other hand, trajectories have trajectories parallel to the direction of saddle connections. Therefore, they cover the whole surface of $L$.

Furthermore, if a saddle connection of width $\varepsilon$, then there is a trajectory perpendicular to $L$ of two saddle connections, or there are areas of certain $\omega$-triangles, such that $\omega$-triangle area.

In view of the development of saddle connections $L$ and $L'$ we have, $L_0 = L, L_1, \ldots, L_k = L'$ in the single pencil. The number of directions parallel to the given direction is $\omega$ only. Consequently, there are not exceed the constant $N$.

We now turn to the second case. Let $\overline{v}$ be the direction of the geodesic flow in the direction of the pencil into pencils of periodic trajectories. The lengths of parallel saddle connections $L$ let $\overline{v}$ be the direction parallel to the given direction.
\subsection*{6. Another proof of Veech's theorem}

\subsection*{6.1. Property A.}

\textbf{Definition 6.1.} Let $\omega$ be a planar structure on a surface $M$. An $\omega$-triangle is a triangle on $M$ whose vertices are singular points, whose sides are saddle connections, and whose interior does not contain singular points.

\textbf{Definition 6.2.} A planar structure $\omega$ is said to have property A if the area of each $\omega$-triangle is larger than a positive constant. $\omega$ is said to have property B if, moreover, the area of each $\omega$-triangle can only assume finitely many values.

\textbf{Proposition 6.1.} Property A is equivalent to the following: if a planar structure $\omega$ has a saddle connection in some direction, then all trajectories in that direction are either periodic or saddle connections; moreover, the ratio of the lengths of parallel saddle connections is bounded above by a constant which does not depend on the direction.

\textbf{Proof.} We assume that $\omega$ has property A, and let $\delta$ be a positive constant smaller than the area of every $\omega$-triangle. We consider an arbitrary saddle connection $L$. Through a point $x \in L$ we draw the geodesic interval $J$ of length $\varepsilon = 2\delta/l$, where $l$ is the length of $L$, perpendicular to $L$. The trajectories emitted from any point of this interval distinct from $x$ and parallel to $L$ do not hit a singular point (for otherwise there would be an $\omega$-triangle with $L$ as one of its sides and with altitude perpendicular to $L$ not exceeding $J$; the area of this $\omega$-triangle would be no larger than $1/2 \cdot l \cdot \varepsilon = \delta$). This implies that all such trajectories are periodic. Thus, an arbitrary saddle connection bounds $2$ (possibly coincident) pencils of periodic trajectories. On the other hand, there are only finitely many pencils of periodic trajectories parallel to the given direction, and each such pencil is bounded by saddle connections. Therefore the saddle connections and periodic pencils parallel to $L$ cover the whole surface $M$, since $M$ is connected.

Furthermore, if a saddle connection $L$ bounds a pencil of periodic trajectories of width $\varepsilon$, then there is an $\omega$-triangle with $L$ as one of its sides and whose altitude perpendicular to $L$ is equal to $\varepsilon$. This implies that the ratio of the lengths of two saddle connections bounding the same pencil is equal to the ratio of the areas of certain $\omega$-triangles, and therefore does not exceed $S/\delta$, where $S$ is the area of $M$. In view of the assertions proved above, for any pair of parallel saddle connections $L$ and $L'$ we can find a chain of pairwise distinct saddle connections $L_0 = L, L_1, \ldots, L_k = L'$ in which any pair of adjacent saddle connections bounds a single pencil. The number of saddle connections and the number of periodic pencils parallel to the given direction do not exceed a certain number $k_0$ which depends on $\omega$ only. Consequently, the ratio of the lengths of parallel saddle connections does not exceed the constant $(S/\delta)^{k_0-1}$, as required.

We now turn to the second part of the assertion. We assume that $\omega$ is such that the geodesic flow in the direction parallel to the saddle connection splits completely into pencils of periodic trajectories and, moreover, we assume that the ratio of the lengths of parallel saddle connections is bounded above by a constant $C$. Further, let $\overline{v}$ be the direction parallel to the saddle connection, and $P$ an arbitrary pencil
of periodic trajectories in this direction. We choose the saddle connection $L$ that intersects the trajectories of $P$ and does not leave its boundaries (the end-points of $L$ lie on oppositely located sides of the pencil). Let $l$ be the length of $L$, and let $\alpha$ be the angle between $L$ and the trajectories in the pencil. Then the width of $P$ is $l \sin \alpha$. Furthermore, let $P_1$ be another pencil of periodic trajectories in the direction $\bar{v}$. Since all trajectories parallel to $L_1$ are either periodic or saddle connections, we can find a saddle connection $L_1$ parallel to $L$ and intersecting the trajectories of $P_1$ (in general, $L_1$ goes beyond the boundary of $P_1$). If $l_1$ denotes the length of $L_1$, then the width of $P_1$ and $P$ is at most $\frac{k_1}{l \sin \alpha} = \frac{k_1}{l} \leq C$. The ratio of their lengths does not exceed $Ck_1$, where $k_1$ is the number of saddle connections in the direction $\bar{v}$. Consequently, the ratio of the areas of $P_1$ and $P$ is at most $C^2k_1$. We denote by $k_2$ the number of pencils of periodic trajectories parallel to $\bar{v}$. The total sum of the areas of all pencils is $S$, the area of $M$, so that the area of each pencil is at least $\frac{S}{1+C^2k_2k_0(k_0-1)}$. As already noted above, $k_1$ and $k_2$ are bounded above by the constant $k_0$, which does not depend on the direction $\bar{v}$. Consequently, the area of an arbitrary pencil of periodic trajectories is at least equal to the constant $\frac{S}{1+C^2k_0(k_0-1)}$.

We now consider an arbitrary $\omega$-triangle $T$. Let $L$ be a side of it and let $P$ be the pencil of periodic trajectories bounded by the saddle connection $L$ and lying on the same side of $L$ as $T$. Clearly, the width of $P$ is not larger than the altitude of $T$ perpendicular to $L$, and its length is not larger than $Ck_0$ times the length of $L$. So, the ratio of the area of the pencil to the area of the $\omega$-triangle is at most $2Ck_0$, and hence the area of the $\omega$-triangle is at least equal to the constant $\frac{S}{1+C^2k_0(k_0-1)} \cdot \frac{l}{2k_0} > 0$. Thus, the $\omega$-triangle has the property A.

Proposition 6.2. Suppose that the planar structure $\omega$ has property A. Then the area of each pencil of periodic trajectories is larger than a positive constant. If $r_1, r_2, \ldots, r_k$ are the ratios of the length to the width for all pencils of periodic trajectories in a certain direction (parallel to a saddle connection), then $r_1, r_2, \ldots, r_k$ are rationally commensurate and the ratio $\frac{\mathrm{LCM}(r_1, r_2, \ldots, r_k)}{r_i}$ is bounded above by a constant which does not depend on the direction. (See §3.2 for the definition of $\mathrm{LCM}(r_1, r_2, \ldots, r_k).$)

Proof. The fact that the area of each pencil of periodic trajectories is larger than a positive constant $S_0$ is obvious, since each pencil contains an $\omega$-triangle. Let $P_1$ and $P_2$ be two pencils of periodic trajectories bounded by a single saddle connection $L$ of length $l$, and let $r_1$ and $r_2$ be the ratios of the length to the width of each of them. We will show that $r_1$ and $r_2$ are commensurate and that their least common multiple exceeds them by at most $k_0S_0$ times, where $S_0$ is the area of $M$ and $k_0$ is the maximum possible number of distinct saddle connections or periodic pencils in a single direction.

Without loss of generality we may assume that $P_1$ and $P_2$ are horizontal. Let $L_0$ be the saddle connection intersecting the trajectories of $P_1$, not going beyond the boundaries of $P_1$ and $P_2$. Clearly, each $\omega$-triangle is contained in the saddle connection $L_0$, and for any $n \in \mathbb{Z}$ a saddle connection $h^nL_0$, $h^{n+1}L_0$, $\omega$-triangle $T_n$. Let $l_n$ be the length of $L_n$, and let $Q_n$ be the orthogonally projective pencil containing $\omega$-triangle $T_n$; it is clearly an $h^{n+1}r_0$, $\omega$-triangle parallel to $L_n$, while the altitude from $Q_n$ to $L_n$ is the length of the $\omega$-triangle $T_n$. Since $l_n/l < S_0/S$, it is larger than $S_0/S$.

The distance between the points $Q_n$ is at least the length $l_n$ of the pencil, is, and it follows that $r_1$ and $r_2$ are everywhere dense on $L$. The points $Q_n$ divide the sides into subsegments of length $l_n/l > S_0/S$. Since $l_n/l < S_0/S$, the subsegments are arbitrarily close to $l_n/l > S_0/S$.

The similar statement holds for $r_2/2k_0$. The same reasoning.

All pencils of periodic trajectories parallel to the subsegments of $L$, each containing an $\omega$-triangle with $r_1, r_2, \ldots, r_k$ are the ratios of the lengths of the pencil $P_k$ to the area $S_0$. (This was proved above, by induction on $k$, using the fact that $C^2k_0/S_0$ is a positive constant.) The constant $C^2k_0/S_0$ is independent of $k$.

Taking into account Lemma 2.3, we can find a vector $\bar{v}$ parallel to a saddle connection such that $a\bar{v} = \bar{v}$. Thus, $\bar{v}$ is parallel to $L_0$.

Proposition 6.3. Suppose that the vector $\bar{v}$ is not parallel to any saddle connection, then the $\omega$-triangle $\bar{v}$ is strongly ergodic.

Proof. We prove this by contradiction. Suppose that the flow is not strongly ergodic. Then we must find an $l_0 = l_0(\varepsilon)$ such that...
the saddle connection $L$ that has its boundaries (the end-points). Let $l$ be the length of $L$, and in the pencil. Then the width of any periodic trajectories in $P_1$ are either periodic or saddle parallel to $L$ and intersecting the boundary of $P_1$). If $l_1$ denotes $l_1 \sin \alpha$, so that the ratio of the The ratio of their lengths does connections in the direction $\overline{v}$. is at most $C^2 k_1$. We denote $\frac{v}{l_1}$ parallel to $\overline{v}$. The total sum of the area of each pencil $k_1$ and $k_2$ are bounded above direction $\overline{v}$. Consequently, the $s$ at least equal to the constant $L$ be a side of it and let $P$ be a saddle connection $L$ and lying $\frac{1}{r}$ is not larger than the altitude $C k_0$ times the length he area of the $\omega$-triangle is at least equal to the constant $\frac{1}{r}$ property $A$.

$\omega$ has property $A$. Then the $\frac{1}{r}$ than a positive constant. If $h$ for all pencils of periodic trajectories), then $r_1, r_2, \ldots, r_k \frac{r_2}{r_1}, \ldots, \frac{r_k}{r_1}$ is bounded above by $r_1$. (See §3.2 for the definition of $r_1$. dig trajectories is larger than a $\omega$-triangle. Let $P_1$ and $\gamma$ a single saddle connection $L$ length to the width for each of the least common area $S$ is the area of $M$ and $k_0$ connections or periodic pencils $P_1$ and $P_2$ are horizontal. Let trajectories of $P_1$, not going beyond the boundaries of $P_1$ and having common end-point $O$ with $L$. We put

$$h_r = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{for all} \quad x \in \mathbb{R}.$$ Clearly, each $\omega$-triangle is also an $h_r \omega$-triangle, with the same area. For some $a \in \mathbb{R}$ the saddle connection $L_0$ is vertical with respect to $h_a \omega$. In this case we can find for any $n \in \mathbb{Z}$ a saddle connection $L_n$ intersecting $P_1$ that goes out from $O$ and is vertical with respect to $h_{a+nr_1} \omega$. Moreover, $L$ and $L_n$ are legs of the right-angled $h_{a+nr_1} \omega$-triangle $T_n$. Let $Q$ be the singular point on the side of $P_2$ lying opposite $L$, and let $Q_n$ be the orthogonal (with respect to $h_{a+nr_1} \omega$) projection of $Q$ on the side of the pencil containing $L$. If for some $n \in \mathbb{Z}$ the point $Q_n$ lies on $L$, then there is clearly an $h_{a+nr_1} \omega$-triangle $T$ with side $L_n$ such that $Q$ is the vertex opposite to $L_n$, while the altitude from $Q$ is equal to the interval $Q_nO$. In this case the ratio of the length of $Q_nO$ to $l$ is equal to the ratio of the areas of $T$ and $L_n$, and hence it is larger than $S_0/S$.

The distance between points $Q_n$ and $Q_{n+1}$ on the side of $P_2$, divided by the length $l_2$ of the pencil, is, up to an integer factor, equal to $r_1/r_2$. This immediately implies that $r_1$ and $r_2$ are commensurate, since otherwise the points $Q_n$ would be everywhere dense on $L$. Let $r$ be the least common multiple of $r_1$ and $r_2$. The points $Q_n$ divide the side of $P_2$ into $r/r_1$ equal parts. Now, if $l' = l_2 r_1/r$ does not exceed the length $l$ of $L$, then there exists an $n \in \mathbb{Z}$ such that $Q_n$ belongs to $L$ and the length of $Q_nO$ does not exceed $l'$. In view of what was said above, $l'/l > S_0/S$. Since $l_2/l < k_0$, we arrive at the required estimate $r/r_1 < k_0 S/S_0$. The similar estimate for $r/r_2$ can be obtained by interchanging $P_1$ and $P_2$ in the above reasoning.

All pencils of periodic trajectories in a single direction can be arranged in a sequence $P_1, P_2, \ldots, P_k$, where each pencil, from $P_2$ onwards, is adjacent (that is, has a common saddle connection on its boundary) to some previous one. Let $r_1, \ldots, r_k$ be the ratios of the length to the width of $P_1, \ldots, P_k$. Using what was proved above, by induction with respect to $i$ we find that the least common multiple of the numbers $r_1, \ldots, r_i$ ($1 \leq i \leq k$) exceeds each of these numbers by at most a factor $C^{i-1}$, $C = k_0 S/S_0$. Since $k \leq k_0$, we have $\frac{\text{LCM}(r_1, \ldots, r_k)}{r_i} < C^{k_0-1}$, where the constant $C^{k_0-1}$ is independent of the direction.

Taking into account Lemma 3.9, Propositions 6.1 and 6.2 imply that for any vector $\overline{v}$ parallel to a saddle connection, the stabilizer $\Gamma(\omega)$ contains an element $a$ such that $a \overline{v} = \overline{v}$. Thus, $\Gamma(\omega)$ is a sufficiently rich group.

**Proposition 6.3.** Suppose that the planar structure $\omega$ has property $A$. Let $\overline{v}$ be a vector not parallel to any saddle connection. Then the geodesic flow on $M$ in the direction $\overline{v}$ is strongly ergodic.

**Proof.** We prove this by contradiction. We assume that the geodesic flow on $M$ in the direction $\overline{v}$ is not strongly ergodic. Since $\overline{v}$ is not parallel to a saddle connection, this flow must be minimal, hence we can use Theorem 3.11. So, for $\varepsilon > 0$ we can find an $l_0 = l_0(\varepsilon)$ such that for all $l \geq l_0$ there is a saddle connection $L_{l,\varepsilon}$ of length
at most \( l \) and whose projection on the direction \( \vec{u} \) perpendicular to \( \vec{v} \) has length less than \( \varepsilon/l \). Let \( L' \) be the shortest saddle connection that we can take for \( L_{0,\varepsilon} \), and let \( h \) be its projection on the direction \( \vec{u} \) (\( h < \varepsilon/l_0 \)). Let \( L'' \) be the saddle connection \( L_{1,\varepsilon} \), where \( l_1 = \varepsilon/h \). Its projection on the direction \( \vec{u} \) is smaller than \( \varepsilon/l_1 = h \), so, by the choice of \( L' \), the saddle connection \( L'' \) is not shorter than \( L' \). This implies that they are not parallel.

Let \( l', l'' \) be the lengths of \( L' \), \( L'' \), let \( \alpha', \alpha'' \) be the angles they form with the direction \( \vec{v} \), and let \( \alpha \) be the angle between the saddle connections themselves (\( \alpha, \alpha', \alpha'' \in [0, \pi/2] \)). We have: \( l_1 \geq l'' \geq l' \), \( 0 < \alpha \leq \alpha' + \alpha'' \). By construction, \( \sin \alpha' = h/l' \), \( \sin \alpha'' = h/l'' \). On the other hand, the projection of \( L'' \) on the direction perpendicular to \( L' \) is at least \( w' \) (the width of some pencil of periodic trajectories parallel to \( L' \)). The proof of Proposition 6.1 implies that \( w' \geq C/l' \), where \( C \) is a constant depending on the planar structure only. As a result we find that \( \sin \alpha \geq w'/(l' + l'') \geq \frac{C}{l' + l''} \). Since \( \frac{\varepsilon}{\pi} \beta \leq \sin \beta \leq \beta \) for \( 0 \leq \beta \leq \pi/2 \), we obtain:

\[
\frac{C}{l' + l''} \leq \sin \alpha \leq \alpha' + \alpha'' \leq \frac{2}{\pi} \left( \frac{C}{l' + l''} \right)
\]

whence

\[
C < \frac{2h}{\pi} \left( \frac{l' + l''}{l_1} \right) = \frac{2\varepsilon}{\pi} \left( \frac{l' + l''}{l_1} \right) \leq \frac{4\varepsilon}{\pi}.
\]

Thus, \( \varepsilon > \frac{\pi}{4} C \), that is, \( \varepsilon \) cannot be made arbitrarily small, contradicting the assumption. This proves the proposition.

We summarize the results obtained in this subsection.

**Proposition 6.4.** A planar structure having property A satisfies the Veech alternative.

So, from the point of view of the behaviour of geodesic flows, planar structures having property A do not differ from planar structures having a lattice stabilizer. A difference between them will be established in the next subsection.

### 6.2. Property B

We will first prove certain assertions concerning lattices in \( \text{SL}(2, \mathbb{R}) \) that will be used in the proof of Theorem 6.8.

**Definition 6.3.** A non-zero vector \( \vec{v} \in \mathbb{R}^2 \) is called a parabolic vector of a discrete subgroup \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) if \( a\vec{v} = \vec{v} \) for some \( a \in \Gamma \), \( a \neq 1 \).

The subgroup \( \Gamma \) acts in a natural way on the plane \( \mathbb{R}^2 \). We denote by \( \Gamma \vec{v} \) the orbit of a vector \( \vec{v} \neq 0 \) under this action.

**Lemma 6.5.** If \( \vec{v} \) is a parabolic vector for \( \Gamma \), then the orbit \( \Gamma \vec{v} \) is discrete.

**Proof.** So, \( a\vec{v} = \vec{v} \) for some \( a \in \Gamma \), \( a \neq 1 \). We assume that \( \gamma_i \vec{v} \to \vec{w} \) as \( i \to \infty \) for a sequence \( \{\gamma_i\} \subset \Gamma \) and some \( \vec{w} \in \mathbb{R}^2 \). We put \( a_i = \gamma_i a_0^{-1} \), \( \vec{v}_i = \gamma_i \vec{v} \), \( \vec{u}_i = \gamma_i \vec{u} \), \( \vec{u} \) is a vector orthogonal to \( \vec{v} \). We find that \( a_i \vec{v}_i = \vec{v}_i \), \( a_i \vec{u}_i = \vec{u}_i + a_i \vec{u}_i \), \( i \in \mathbb{R} \), where \( a_i \vec{v}_i \) are bounded above and all \( \gamma_i \vec{v}_i \) preserve area, the length of the projection of \( \vec{u}_i \) on the direction orthogonal to \( \vec{v}_i \) is bounded below by a positive constant. This implies that the sequence \( \{a_i\} \) is bounded in \( \text{SL}(2, \mathbb{R}) \). Since \( \Gamma \) is discrete, there are only finitely many distinct \( a_i \). Without loss of generality we may assume that all the vectors \( \vec{v}_i \) are collinear and that \( \lambda_i \neq 0 \). If \( |\lambda_i| < 1 \), the sequence \( \{\gamma_i \vec{v}_i\} \) tends as \( n \to +\infty \); if \( |\lambda_i| > 1 \), the sequence tends to a discrete set of \( \gamma \). Hence, for \( n \to +\infty \), coincides up to multiplicity with \( \vec{v}_1, \ldots, \vec{v}_k \) are the eigenvectors of these orbits are discrete.

Now, let \( \vec{v} \) be a non-zero vector; assume that \( \vec{v} \) is vertical. Then \( \vec{v} \) is a vector; then \( \vec{v}_1, \ldots, \vec{v}_k \) are the eigenvectors of these orbits are discrete.

**Lemma 6.6.** Let \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) be a discrete group, and \( \vec{v} \in \mathbb{R}^2 \) is a vector. Then \( \Gamma \vec{v} \) is discrete.

**Proof.** Let \( \vec{u} \) be a parabolic vector, \( \vec{v} \in \mathbb{R}^2 \) is a vertical vector; then \( \vec{v}_1, \ldots, \vec{v}_k \) are the eigenvectors of these orbits are discrete.

### 6.3. Preliminary results

**Lemma 6.7.** A discrete group \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) is a group.

**Proof.** Let \( \vec{u} \) be a lattice vector, then \( \vec{u} \) is a discrete set of \( \gamma \). Hence, \( \vec{u} \) is a vector. Since \( e^{\pi i/2} \vec{u} \) is a vector, moreover, all its

### 6.4. Properties of \( \text{SL}(2, \mathbb{R}) \)

**Lemma 6.8.** A discrete group \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) is a group.

**Proof.** Let \( \vec{v} \) be a lattice vector, then \( \vec{v} \) is a discrete set of \( \gamma \). Hence, \( \vec{v} \) is a vector. Since \( e^{\pi i/2} \vec{v} \) is a vector, moreover, all its
perpendicular to \( \overline{v} \) has length 
\( \varepsilon / \lambda_i \). Let \( L'' \) be the saddle 
the direction \( \overline{u} \) is smaller than 
section \( L'' \) is not shorter than \( L' \).

be the angles they form with 
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\( \alpha \leq \alpha' + \alpha'' \). By construction, 
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icture only. As a result we find 
\( \beta \geq \pi / 2 \), we obtain:

\[
\alpha'' < \frac{2h}{\pi} \left( \frac{1}{l'} + \frac{1}{l''} \right).
\]

\( \gamma \) \leq \frac{4}{\pi}.

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me that \( \gamma_i \overline{v} \to \overline{w} \) as \( i \to \infty \) for 
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\( \overline{v}_i = \overline{v}_i, a \overline{v}_i = \overline{u}_i + \alpha \overline{v}_i, \alpha \in \mathbb{R} \).

and we all \( \gamma_i \) preserve area, the 
onal to \( \overline{v}_i \) is bounded below by a 
| \( i \) is bounded in \( \mathbb{SL}(2, \mathbb{R}) \). Since 
\( \alpha_i \). Without loss of generality

we may assume that all \( a_i \) coincide. Since \( a_i \neq 1 \) and \( a_i \overline{v}_i = \overline{v}_i \), we find that 
the vectors \( \overline{v}_i \) are collinear. We put 
\( b_i = \gamma_i \gamma_i^{-1} \), then 
\( b_i \overline{v}_i = \overline{v}_{i+1} = \lambda_i \overline{v}_i \), with 
\( \lambda \neq 1 \). If \( |\lambda| < 1 \), the sequence of operators 
\( b_i \gamma_i \overrightarrow{a_i} \) tends to the identity operator 
as \( n \to +\infty \); if \( |\lambda| > 1 \), we have to take 
\( n \to -\infty \). Both cases contradict the 
discreteness of \( \Gamma \), hence \( |\lambda| = 1 \). Then all terms of the sequence \( \{\gamma_i \overline{v}_i\} \), 
from some term onwards, coincide (with \( \overline{w} \)), which proves that 
\( \Gamma \overline{w} \) is discrete.

**Lemma 6.6.** Let \( \Gamma \) be a lattice in \( \mathbb{SL}(2, \mathbb{R}) \). If \( \overline{v} \) is a parabolic vector for \( \Gamma \), then 
the orbit \( \Gamma \overline{v} \) is discrete, otherwise \( 0 \) is a limit point of it. The number of discrete 
ors, considered up to multiplication by a scalar, is finite.

**Proof.** Let \( \overline{v} \) be a parabolic vector for \( \Gamma \), that is, \( a \overline{v} = \overline{v} \) for some \( a \in \Gamma \), \( a \neq 1 \). 
Since \( \Gamma \) is a lattice, \( a \) is conjugate to a power of one of a finite number of elements 
\( a_1, \ldots, a_k \in \Gamma \setminus \{1\} \) (see [16]). So, 
\( \gamma a \gamma^{-1} = a \gamma \) for some \( \gamma \in \Gamma \), \( n \in \mathbb{Z} \) and some \( i \), 
\( 1 \leq i \leq k \). Then \( \gamma \overline{v} \in \Gamma \overline{w} \) is an eigenvector of \( a_i \). So, the orbit of a parabolic vector 
coincides, up to multiplication by a scalar, with one of the operators 
\( \Gamma \overline{v}_1, \ldots, \Gamma \overline{v}_k \), where 
\( \overline{v}_1, \ldots, \overline{v}_k \) are the eigenvectors of the operators 
\( a_1, \ldots, a_k \). Lemma 6.5 asserts that 
these orbits are discrete.

ow, let \( \overline{v} \) be a non-parabolic vector for \( \Gamma \). Without loss of generality we may 
assume that \( \overline{v} \) is vertical (in fact, let \( b \in \mathbb{SL}(2, \mathbb{R}) \) be the operator mapping \( \overline{v} \) 
for a vertical vector; then \( b \overline{v} \) is a non-parabolic vector for the lattice 
\( \Gamma_1 = b \mathbb{SL}(2, \mathbb{R}) \), and \( \Gamma_1 = b \overline{w} \)). In this case, by Lemma 3.7, 
\( g_i \Gamma_1 \to h \) as \( i \to \infty \), where 
\( \{t_i\} \) is some sequence of numbers tending to 
\( +\infty \), \( g_i = \begin{pmatrix} e^{t_i/2} & 0 \\
0 & e^{-t_i/2} \end{pmatrix} \), \( \{\gamma_i\} \subset \Gamma \), 
h \in \mathbb{SL}(2, \mathbb{R}) \). Hence, \( \gamma_i^{-1} g_i^{-t_i} \overline{v} \to h \overline{w} \) as \( i \to \infty \). The vector 
\( \overline{v} \) is vertical, so 
\( g_i^{-t_i} \overline{v} = e^{t_i/2} \overline{v} \). Since \( e^{t_i/2} \to \infty \) as \( i \to \infty \), the sequence 
\( \{\gamma_i^{-1} \overline{v}\} \) tends to the zero 
vector. Moreover, all its terms belong to the orbit \( \Gamma \overline{v} \).

**Lemma 6.7.** A discrete subgroup \( \Gamma \subset \mathbb{SL}(2, \mathbb{R}) \) is a lattice if and only if we can 
choose finitely many parabolic vectors \( \overline{v}_1, \ldots, \overline{v}_k \) in it and for each \( a \in \mathbb{SL}(2, \mathbb{R}) \) 
we can find a vector \( \overline{v} \in \Gamma \overline{v}_i, 1 \leq i \leq k, \) such that 
\( |a \overline{v}| \leq c \), where \( c \) is a constant 
independent of \( a \).

**Proof.** Let \( \Gamma \) be a lattice in \( \mathbb{SL}(2, \mathbb{R}) \). We will use the notations introduced in the 
proof of Lemma 3.7. Let \( \overline{v}_i (1 \leq i \leq k) \) be the image of a horizontal vector under 
the action of the rotation \( r_i \). Then \( a_i \overline{v}_i = \pm a_i \overline{v}_i, a_i \overline{v}_i = \overline{v}_i, \) and \( a_i^2 \in \Gamma, a_i^2 \neq 1 \), that is, 
\( \overline{v}_i \) is a parabolic vector for \( \Gamma \).

We fix a point \( z_0 \in \mathbb{H}^2 \). For any \( a \in \mathbb{SL}(2, \mathbb{R}) \) we can find a \( \gamma \in \Gamma \) such that 
the point \( (a)z_0 \) belongs to the fundamental polygon \( D \). Then \( (a)z_0 \) belongs 
to the compact set \( K \) or to a wedge \( r_i(\Pi(c_i; \alpha, \beta)) \). In the latter case a power of 
\( \alpha_i \) maps the point \( (a)z_0 \) into the wedge \( r_i(\Pi(c_i; \alpha, \beta - \alpha)) \). Any point 
of this wedge can be mapped into the compact set \( r_i(\Pi(c_i; \alpha, \beta - \alpha)) \) by means of 
the operator \( g_i^{-t} = r_i^{-1} g_i^{-t} r_i^{-1} \), where \( t \geq 0 \) depends on the point. Thus, for any 
\( a \in \mathbb{SL}(2, \mathbb{R}) \) we can find a \( \gamma \in \Gamma \), an index \( i, 1 \leq i \leq k \), and a number \( t \geq 0 \) 
such that \( (g_i^{-t} \gamma)z_0 \) belongs to a compact set \( K \subset \mathbb{H}^2 \) which does not depend 
on \( a \). Consequently, \( a = \gamma^{-1} g_i^s \), where \( s \) belongs to the set 
\( K \subset \mathbb{SL}(2, \mathbb{R}) \) of operators mapping \( z_0 \) into \( K \) (clearly, \( K_1 \) is compact). Replacing \( a \) by \( a^{-1} \) in the
above reasoning, we find that \( a \) can be written in the form \( a = s^{-1} q_i^{-t} \), with \( \gamma \in \Gamma \), \( s \in K_1 \), \( 1 \leq i \leq k \), \( t \geq 0 \). We put \( \vartheta = \gamma^{-1} s \). Then \( \vartheta \in \Gamma \) and 
\[
\langle a \vartheta \rangle = s^{-1} q_i^{-t} \langle \vartheta \rangle = s^{-1} (e^{-t/2} \langle \vartheta \rangle) = e^{-t/2} (s^{-1} \langle \vartheta \rangle).
\]
Since \( s \) lies in a compact subset of \( SL(2, \mathbb{R}) \), the quantity \( |s^{-1} \langle \vartheta \rangle| \) is bounded by a constant \( C \) which does not depend on \( s \) and \( i \), that is, \( |a \vartheta| = e^{-t/2} |s^{-1} \langle \vartheta \rangle| \leq e^{-t/2} \cdot C \leq C \). This proves one assertion in the lemma.

We now assume that \( \Gamma \) is a discrete subgroup of \( SL(2, \mathbb{R}) \) for which we can choose finitely many parabolic vectors \( \vartheta_1, \ldots, \vartheta_k \) in accordance with the conditions of the lemma. Let \( a_1, \ldots, a_k \) be non-identity operators in \( \Gamma \) such that \( a_i \vartheta_i = \vartheta_i, 1 \leq i \leq k \). Let \( r_i (1 \leq i \leq k) \) be the rotation operators mapping the horizontal vector to \( \vartheta_i \); then 
\[
\alpha_i r_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}, \quad \alpha_i \neq 0.
\]
By requirement, for each \( a \in SL(2, \mathbb{R}) \) we can find a \( \alpha \in \Gamma \) and an index \( i \), \( 1 \leq i \leq k \), such that \( |a \alpha \vartheta_i| \leq C \), where \( C \) is a constant. We denote by \( h_\alpha \) a vector orthogonal to \( \vartheta_i \), and of the same length. Clearly, \( a_i h_\alpha = \vartheta_i \pm \alpha_i h_\alpha \). For some \( n \in \mathbb{Z} \) the length of the projection of the vector \( a_i \alpha \vartheta_i \) on the direction of the vector \( a_i \vartheta_i \) does not exceed \( |a_i| \cdot |a \vartheta_i| \). We put 
\[
g_i = r_i \vartheta_i^{-1} r_i^{-1};
\]
then, 
\[
g_i \vartheta_i = e^{-t/2} \vartheta_i, g_i \vartheta_i = e^{-t/2} \vartheta_i,
\]
and for some \( t \geq 0 \) we have 
\[
|a_i \alpha \vartheta_i| = C.\]
Moreover, the length of the projection of \( a_i \alpha \vartheta_i \) on the direction of the vector \( a_i \vartheta_i \) does not exceed \( e^{-t/2}; \max |a_i| \cdot |a \vartheta_i| \leq C \). On the other hand, the length of its projection on the orthogonal direction is \( \| \vartheta_i \| / \| h_\alpha \| = \| \vartheta_i \| / C \), since \( a_i \alpha \vartheta_i \) belongs to \( SL(2, \mathbb{R}) \). Thus, \( |a_i \alpha \vartheta_i| \leq \left( |a_i|^2 C^2 + \| h_\alpha \|^2 / C^2 \right)^{1/2} \), which does not exceed some constant. Consequently, \( a_i \alpha \vartheta_i \) belongs to a compact subset of \( SL(2, \mathbb{R}) \).

What was said above implies that each \( a \in SL(2, \mathbb{R}) \) can be written as \( a = \gamma g_i s \), with \( \gamma \in \Gamma \), \( 1 \leq i \leq k \), \( t \geq 0 \), and \( s \) belonging to the compact set \( K \subset SL(2, \mathbb{R}) \). We fix a point \( z_0 \in \mathbb{H}^2 \). For any \( s \in K_1 \) the point \( s z_0 \) belongs to the compact set \( K \subset \mathbb{H}^2 \), which does not depend on \( s \). We now choose \( c > 0 \) so small that \( K \subset \bigcap_{i=1}^{k} r_i (c) \). Since \( g_i c \in r_i (c) \) for \( t \geq 0 \), if \( z \in r_i (c) \) we find that 
\[
\gamma^{-1} (a z_0) = g_i (s z_0) \in r_i (c).
\]
An arbitrary point \( z \in \mathbb{H}^2 \) can be written as \( a z_0 \), \( a \in SL(2, \mathbb{R}) \), so, under the action of an element of \( \Gamma \) it is mapped into one of the sets \( r_i (c) \). Applying now some power of \( a \), we can map this point into the wedge \( r_i (c) \). The non-Euclidean area of each wedge is finite (see [16]). In view of the discreteness of \( \Gamma \) there is thus a fundamental domain of finite area for its action on \( \mathbb{H}^2 \), that is, \( \Gamma \) is a lattice.

**Theorem 6.8.** A planar structure has property B if and only if its stabilizer is a lattice.

**Proof.** The group of affine automorphisms of the planar structure \( \omega \) acts in a natural manner on the set of saddle connections and on the set of \( \omega \)-triangles.

**Lemma 6.9.** If the stabilizer \( \Gamma(\omega) \) of a planar structure \( \omega \) is a lattice, then the number of orbits of the action on each of the sets listed above is finite.

**Proof.** Let \( \vartheta \) be a vector in \( \mathbb{R}^2 \) that is a development of a saddle connection. Then, for any \( \alpha \in \Gamma(\omega) \) the vector \( \alpha \vartheta \) is also a development of a saddle connection. Since there do not exist arbitrarily short saddle connections, \( 0 \) cannot be a limit point for the orbit \( \Gamma(\omega) \).

In addition we can choose \( \vartheta \) so that the orbit \( \Gamma(\omega) \vartheta \) contains no type of degeneration. In particular, for each \( s \) and \( i \), the segment parallel to the velocity field \( \vartheta_i \) which crosses \( \Gamma(\omega) \vartheta \) does not intersect the segment parallel to the velocity field \( \vartheta_i \) which crosses \( \Gamma(\omega) \vartheta \).

We now consider an arbitrary planar structure it can be decomposed into segments \( L \) denoted by \( L \), parallel to the velocity field \( \vartheta_i \) for which \( a_i \vartheta_i = \vartheta_i \). Applying the linear part \( a_i \), we obtain an orbit of \( \Gamma(\omega) \) whose length at this side is larger than \( C \).

Since the area of \( T_2 \) is bounded from below by the area of the sides of \( T_2 \) are bounded, our planar structure contains finitely many saddle connections. Hence, there are only finitely many saddle connections.

Since the area of an \( \omega \)-triangles in the above lemma it is clear that \( \omega \) has the property B.

We now turn to the case where the planar structure is not parallel to a saddle connection and splits into finitely many pencils of these pencils.

**Lemma 6.10.** The sequence \( \vartheta \) is determined by a scalar, with one of the segments.

**Proof.** If the pencils \( P_1, \ldots, P_k \) have the same boundary, then we can take only finitely many pencils of \( \omega \) and each intersection of these pencils is in such a way that each pencil intersects one of the previous pencils at a point \( w_1, \ldots, w_k \) coincides, up to sequences. To finish the proof we need to rewrite it bounded by a constant and using\( k \).

We now consider a pencil \( P_i \) of the planar structure \( \omega \) that is parallel to \( \vartheta_1 \) and \( \vartheta_2 \) intersect, \( i = 1, \ldots, k \). Let \( p_j \) be right-, left-, up-, or down-right of \( p_i \) by moving along the segments \( p_1, \ldots, p_k \).

Let \( \Pi \) and \( \Pi' \) be the two sets of directions \( (\vartheta_1', \vartheta_2') \) and \( (\vartheta_1'', \vartheta_2'') \). Consider the bijection \( f: \Pi \to \Pi' \) such that \( f(p_j) \) is adjacent to \( f(p_i) \) if \( p_i \) is a saddle connection.
the form \( a = s^{-1}g_i^{-1} \gamma_i \), with \( \gamma = \gamma_i^{-1} \).

Then \( \bar{v} \in \Gamma_i \) and \( \bar{v} \) lies in a compact subset of \( C \) which does not depend on \( \gamma \). This proves one assertion.

\[ \text{L}(2, \mathbb{R}) \] for which we can choose \( a_1, \ldots, a_k \) such that \( a_i \bar{v}_i = \bar{v}_i, 1 \leq i \leq k \).

\( \text{ment for each } a_i \in \text{SL}(2, \mathbb{R}) \) we have \( |a_i \gamma \bar{v}_i| \leq C \), where \( C \) is a constant.

\( \text{h} \) of the projection of the vector \( \bar{v} \) does not exceed \( |a_i| \cdot |\gamma \bar{v}_i| \).

\( \text{R} \) can be written as \( a = \gamma g_i s \). The compact set \( \mathbb{K}_s \subset \text{SL}(2, \mathbb{R}) \).

If \( \alpha \gamma g_i s \) belongs to the compact form of \( \bar{v} \) choose \( \alpha > 0 \) so small that \( \alpha \gamma g_i s \) is mapped into one of the \( \alpha \gamma g_i s \) can be written as \( \alpha \gamma g_i s \).

\( \Gamma \) it is mapped into one of the \( \Gamma \) map this point into the wedge is finite (see [16]). In view of the domain of finite area for its

\[ \text{if and only if its stabilizer is a} \]

\[ \text{nar structure } \omega \text{ acts in a natural way on the set of } \omega \text{-triangles.} \]

\[ \text{structure } \omega \text{ is a lattice, then the set above is finite.} \]

\[ \text{nt of a saddle connection of } \omega \text{.}\]

\[ \text{omment of a saddle connection, mentions, 0 cannot be a limit point for the orbit } \Gamma(\omega) \bar{v}, \text{ and by Lemma 6.6, } a \bar{v} = \bar{v} \text{ for some } a \in \Gamma(\omega), a \neq 1. \]

In addition we can choose finitely many vectors \( \bar{v}_1, \ldots, \bar{v}_n \), independent of \( \bar{v} \), such that the orbit \( \Gamma(\omega) \bar{v} \) contains a vector parallel to one of these. Thus, using an affine automorphism, any saddle connection of \( \omega \) can be mapped to a saddle connection parallel to one of the vectors \( \bar{v}_1, \ldots, \bar{v}_n \). There are only finitely many such saddle connections.

We now consider an arbitrary \( \omega \)-triangle \( T \). By an affine automorphism of the planar structure it can be transformed to an \( \omega \)-triangle \( T_1 \) with one of its sides, denoted by \( L \), parallel to some \( \bar{v}_i \) (1 \( \leq i \leq n \)). Let \( a_i \neq 1 \) be an element of \( \Gamma(\omega) \) for which \( a_i \bar{v}_i = \bar{v}_i \).

Applying to \( T_1 \) some power of the affine automorphism with linear part \( a_i \), we obtain an \( \omega \)-triangle \( T_2 \) with side \( L \) for which one of the angles at this side is larger than a certain constant \( \alpha_1 > 0 \). This constant depends on \( a_i \).

Since the area of \( T_2 \) is bounded (by the area of \( M \)), this implies that the lengths of the sides of \( T_2 \) are bounded by a constant depending on \( L \).

Since \( L \) is one of finitely many saddle connections, an arbitrary \( \omega \)-triangle can be transformed by an affine automorphism to an \( \omega \)-triangle whose side lengths are bounded by a constant.

There are only finitely many such \( \omega \)-triangles.

Since the area of an \( \omega \)-triangle does not change under an affine automorphism, the above lemma immediately implies that a planar structure with a lattice stabilizer has the property \( B \).

We now turn to the second part of the assertion of the theorem. We assume that the planar structure \( \omega \) has the property \( B \). We choose an arbitrary direction \( \bar{v} \) parallel to a saddle connection. By Proposition 6.1, the flow in the direction \( \bar{v} \) splits into finitely many periodic pencils \( P_1, \ldots, P_k \). Let \( w_1, \ldots, w_k \) be the widths of these pencils.

**Lemma 6.10.** The sequence \( w_1, \ldots, w_k \) coincides, up to multiplication of all terms by a scalar, with one of finitely many sequences, independent of \( \bar{v} \).

**Proof.** If the pencils \( P_1 \) and \( P_2 \) are adjacent (have a common saddle connection on their boundary), then \( w_i/w_j \) is the ratio of the areas of certain \( \omega \)-triangles, so it can take only finitely many values. Further, the pencils \( P_1, \ldots, P_k \) can be ordered in such a way that each of them, from the second onwards, is adjacent to at least one of the previous pencils. Hence by induction on \( k \) we find that the sequence \( w_1, \ldots, w_k \) coincides, up to multiplication by a scalar, with one of finitely many sequences.

To finish the proof we need to note that the number \( k \) of pencils is bounded by a constant not depending on \( \bar{v} \).

We now consider a pair of directions \( (\bar{v}_1, \bar{v}_2) \) parallel to saddle connections and giving the standard orientation in \( \mathbb{R}^2 \). The pencils of periodic trajectories parallel to \( \bar{v}_1 \) and \( \bar{v}_2 \) intersect, giving parallelograms. We will say that the parallelogram \( p_1 \) is right-, left-, up-, or down-adjacent to a parallelogram \( p \) if we can go from \( p \) to \( p_1 \) by moving along the vectors \( \bar{v}_1, -\bar{v}_1, \bar{v}_2, \) or \( -\bar{v}_2 \).

Let \( \Pi^+ \) and \( \Pi^- \) be the sets of parallelograms corresponding to the pairs of directions \( (\bar{v}_1', \bar{v}_2') \) and \( (\bar{v}_1'', \bar{v}_2'') \). We call such pairs equivalent if there is a bijection \( f: \Pi^+ \rightarrow \Pi^- \) such that the parallelogram \( f(p_1) \) is right-, left-, up-, or down-adjacent to \( f(p) \) if \( p_1 \) is adjacent to \( p \) with respect to the corresponding side.
The pairs \((\bar{v}_1, \bar{v}_2)\) and \((\bar{v}_1', \bar{v}_2')\) are called \textit{strongly equivalent} if the bijection \(f\) can be chosen such that a parallelogram with sides \(w_1, w_2\) becomes a parallelogram with sides \(\alpha w_1, \beta w_2\), where \(\alpha\) and \(\beta\) are constants.

We assume that a pair of directions \((\bar{v}_1, \bar{v}_2)\) is such that we can find saddle connections \(L_1\) and \(L_2\) parallel to these directions that are two of the sides of an \(\omega\)-triangle. The proof of Proposition 6.1 then implies that the area of each parallelogram corresponding to \((\bar{v}_1, \bar{v}_2)\) is bounded below by a constant \(C > 0\) which depends on \(\omega\) only. This implies that the number of parallelograms does not exceed \(S/C\), where \(S\) is the area of \(M\). Clearly, \((\bar{v}_1, \bar{v}_2)\) is now equivalent to one of finitely many pairs. Furthermore, Lemma 6.10 implies that this pair is also strongly equivalent to finitely many pairs.

The proof that \(\Gamma(\omega)\) is a lattice can now be obtained from Lemma 6.7. In fact, by Lemmas 3.8, 3.9 and Proposition 6.2, the parabolic directions of \(\Gamma(\omega)\) are precisely the directions parallel to saddle connections. Furthermore, for any parabolic direction \(\bar{v}_1\) we can find a direction \(\bar{v}_2\) such that the pair \((\bar{v}_1, \bar{v}_2)\) gives the standard orientation in \(\mathbb{R}^2\) and there exist saddle connections \(L_1\) and \(L_2\) parallel to \(\bar{v}_1\) and \(\bar{v}_2\) that are sides of a single \(\omega\)-triangle. What was said above implies that the pair \((\bar{v}_1, \bar{v}_2)\) is strongly equivalent to some pair \((\bar{v}_1^{(i)}, \bar{v}_2^{(i)})\) from a finite number of pairs \((\bar{v}_1^{(1)}, \bar{v}_2^{(1)}), \ldots, (\bar{v}_1^{(m)}, \bar{v}_2^{(m)})\). The definition of strong equivalence now implies that there is an element \(\gamma \in \Gamma(\omega)\) mapping \(\bar{v}_1\) to \(\bar{v}_1^{(i)}\) and \(\bar{v}_2\) to \(\bar{v}_2^{(i)}\). Thus, the orbit \(\Gamma(\omega)\bar{v}_1\) of a parabolic vector coincides, up to multiplication by a scalar, with one of the orbits \(\Gamma(\omega)\bar{v}_1^{(i)}, 1 \leq i \leq m\). This implies that the sequence \(\text{SC}(\omega)\) consists of finitely many orbits. Using this and Proposition 3.2, on the basis of Lemma 6.7, we may conclude that \(\Gamma(\omega)\) is a lattice.

Since property A is weaker than property B, Theorems 6.4 and 6.8 imply Veech’s theorem.

Proposition 6.2 and Theorem 6.8 indicate that, probably, property A implies property B. It would be interesting to show this. If, however, this conjecture is not true, the construction of a corresponding counterexample (that is, of a planar structure having property A but not property B) would be no less interesting.

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Planar structures and billiards in rational polygons: the Veech alternative

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