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Billiards in Rational Polygons: Periodic Trajectories, Symmetries, and d-Stability

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ABSTRACT. Periodic trajectories of billiards in rational polygons satisfying the Veech alternative, in particular, in right triangles with an acute angle of the form π/n with integer n are considered. The properties under investigation include: symmetry of periodic trajectories, asymptotics of the number of trajectories whose length does not exceed a certain value, stability of periodic billiard trajectories under small deformations of the polygon.

KEY WORDS: billiards in rational polygons, periodic trajectories, Veech alternative.

Introduction

A *billiard flow* in a polygon Q describes the rectilinear motion of a point in Q with a unit speed reflecting in the boundary of the polygon according to the law "the angle of incidence equals the angle of reflection". The phase space of the flow is the direct product of Q and the circle of "unit velocities" S^1 with proper identifications on its boundary implied by the law of reflection. The projection of a phase curve on the polygon is called a *billiard trajectory*. A billiard trajectory that hits a vertex of Q is said to be *singular*, its continuation beyond this vertex, in general, is undefined. A trajectory that starts at a vertex of the polygon and arrives at another vertex (that is, doubly singular) is called a *generalized diagonal*.

In what follows, we assume that Q is a *rational* polygon, that is, the angle between any two of its sides is rationally commensurable with π. In this case, all segments of a fixed billiard trajectory are parallel only to a finite number of directions. As a result, the phase space $Q \times S^1$ falls into invariant surfaces "pasted" from a finite number of copies of the polygon Q (of the form $Q \times \{\overline{v}\}$, $\overline{v} \in S^1$). All of these surfaces except two are homeomorphic to the same connected compact orientable surface M and consistently define a *plane structure* on it, that is, a Riemannian metric of zero curvature with a finite number of conic singular points, the angle at each of them being a multiple of 2π (for details see, for instance, [1]).

Little is known about billiard flows in general polygons. At the same time, billiards in rational polygons are fairly well studied.

In this article we consider a number of questions about periodic billiard trajectories in a rational polygon whose corresponding surface with plane structure has a rich group of affine symmetries (for details see §1). These polygons were studied in [1]. Their main property is the alternative discovered by Veech [2]: each nonsingular trajectory of the billiard is either periodic or equidistributed in the polygon.

We shall consider a number of questions including the symmetry of periodic trajectories, the asymptotics of the number of periodic trajectories of length not greater than R, deformational stability of periodic trajectories in a polygon (that is, stability under small deformations of the polygon). One of the results of this paper (on the deformational instability of all periodic billiard trajectories in a right triangle with an acute angle of the form $\pi/(2n)$, where n is an integer) was announced in [3].

§1. Affine symmetries

Let us refine the definition of a plane structure given in the introduction. We deal with a Riemannian metric on a surface M of curvature zero that has a finite number of singular points, each of which is a conic singularity with an angle of $2\pi m$, where m is a positive integer (the *multiplicity* of the singular point).

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In addition, an arbitrary nonsingular point $x \in M$ has a neighborhood U_x isometric to a domain in $\mathbb{R}^2$. Each isometric embedding of this neighborhood into $\mathbb{R}^2$ specifies a local Cartesian frame in U_x. The frames specified by different embeddings differ from one another by rotation, translation, and, possibly, reflection. Since the holonomy group of the Riemannian metric in question is trivial, we can choose local Cartesian coordinates in the neighborhood of each nonsingular point in such a way that in the intersection of any two neighborhoods the corresponding coordinates only differ by a translation. Such a set of local frames will be called *Cartesian coordinates* on the surface M. In what follows, by a ***plane structure*** we shall understand a Riemannian metric of the form described above supplied with Cartesian coordinates.

By means of Cartesian coordinates, each curve on the surface M can be carried over to $\mathbb{R}^2$, which yields its *development*, a plane curve defined uniquely up to all possible translations in $\mathbb{R}^2$. Further we shall consider periodic geodesics on M and *saddle links*, geodesic segments connecting singular points. Their developments are line segments.

Definition 1.1. An *affine symmetry* of a plane structure ω on a surface M is a homeomorphism of M onto itself that takes singular points into singular points and is an affine mapping in the local Cartesian frame corresponding to ω near each nonsingular point.

For any affine symmetry, its linear (homogeneous) part, a 2×2 matrix with determinant ± 1, is uniquely defined. The group $\Gamma(\omega)$ of linear parts of affine symmetries preserving the orientation of the surface is called the *stabilizer* of the plane structure ω. The stabilizer is a discrete subgroup in $\mathrm{SL}(2, \mathbb{R})$ (see [1, 2]). It is usual to say that the plane structure ω has a large number of affine symmetries if $\Gamma(\omega)$ is a lattice in the group $\mathrm{SL}(2, \mathbb{R})$, that is, if the homogeneous space $\mathrm{SL}(2, \mathbb{R})/\Gamma(\omega)$ is of finite volume.

Suppose that $\Gamma(\omega)$ contains a certain lattice Γ. Since the stabilizer is a discrete subgroup in $\mathrm{SL}(2, \mathbb{R})$, it is also a lattice, so Γ is a subgroup of finite index in $\Gamma(\omega)$. Below we give a sufficient condition for Γ to coincide with the entire stabilizer $\Gamma(\omega)$. As a preliminary, let us introduce a number of definitions. Let $\overline{v} \in S^1$ be a vector parallel to a saddle link of the plane structure ω. According to the Veech alternative (see [1, 2]), all nonsingular geodesics parallel to $\overline{v}$ are periodic and constitute a finite number of bundles separated by saddle links. The set of the lengths of these bundles will be called the *l-sequence* for the direction $\overline{v}$; the set of the ratios of these lengths to the respective bundle widths will be called the *r-sequence* for $\overline{v}$. All elements of the r-sequence are commensurable with one another [1].

Further, recall that one of the important parameters of an arbitrary lattice $\Gamma \in \mathrm{SL}(2, \mathbb{R})$ is the number of its *cusps* (*parabolic vertices*), that is, the greatest number of vertices that a fundamental polygon for the action of Γ on the hyperbolic plane may have on the absolute if no two such vertices are identified by this action (see [4]).

Proposition 1.1. *Suppose that the stabilizer $\Gamma(\omega)$ of a plane structure ω contains a lattice Γ with k cusps. Suppose that there are exactly k directions $\overline{v}_1, \dots, \overline{v}_k \in S^1$ parallel to saddle links of the structure ω such that for each pair of directions $\overline{v}_i$ and $\overline{v}_j$ ($i \neq j$) the corresponding r- and l-sequences are distinct up to permutations of their terms and up to multiplication of all of them by the same factor. Then $\Gamma(\omega) = \Gamma$ if any element $a \in \Gamma(\omega)$ such that $a\overline{v}_1 = \pm\overline{v}_1$ belongs to Γ. The last condition holds, for instance, if the following requirements are satisfied simultaneously*:

(i) *Γ contains an operator b such that $b\overline{v}_1 = \overline{v}_1$ and $b\overline{u}_1 = \overline{u}_1 + r\overline{v}_1$, where $\overline{u}_1$ is a unit vector orthogonal to $\overline{v}_1$ and r is the smallest positive number evenly divisible by each element of the r-sequence for the direction $\overline{v}_1$*;

(ii) *for a certain $l \neq 0$, any permutation of the (nonempty) set of saddle links with development $l\overline{v}_1$ can be realized by affine symmetries of ω with linear parts from Γ*;

(iii) *at least one saddle link with development $l\overline{v}_1$ belongs to the common part of the boundary of two noncongruent periodic bundles, or all these saddle links are taken into themselves by a certain affine symmetry with linear part -1, $-1 \in \Gamma$*;

(iv) *all the saddle links parallel to $\overline{v}_1$ lie in the set U obtained from the surface M after deleting from it those bundles of periodic geodesics parallel to $\overline{v}_1$ whose boundary component contains two saddle links of the same length, and deleting the singular points; in addition, each connected component of the set U contains a saddle link with development $l\overline{v}_1$.*

Proof. We begin with the following remark. Let $\bar{v}$ be a vector parallel to a saddle link of the structure ω, and let γ be an element of $\Gamma(\omega)$. Then the bundles of periodic geodesics parallel to $\bar{v}$ can be taken into the bundles of periodic geodesics parallel to $\gamma\bar{v}$ by an affine symmetry with linear part γ. It follows that the l-sequences for the directions $\bar{v}$ and $\gamma\bar{v}$ coincide up to a permutation and the multiplication of all their terms by the same factor. The r-sequences for $\bar{v}$ and $\gamma\bar{v}$ have the same property.

Since Γ is a lattice in $SL(2,\mathbb{R})$ with k cusps, we can choose k nonzero vectors $\bar{e}_1,\dots,\bar{e}_k$ such that for any vector invariant under the action of a nonidentical element of the lattice there is an element of Γ that takes it to a vector collinear to one of the chosen vectors (see [1, Sec. 3.4]). According to the Veech alternative [1, Theorem 3.4], for the vector $\bar{v}$ parallel to a saddle link, there exists an element $a\in\Gamma(\omega)$, $a\neq 1$, such that $a\bar{v}=\bar{v}$. The group Γ is of finite index in $\Gamma(\omega)$, and therefore only a finite number of cosets $\Gamma, a\Gamma, a^2\Gamma, \dots$ are distinct from one another. This means that $a^n\in\Gamma$ for a certain integer $n>0$. We have $a^n\bar{v}=\bar{v}$ and $a^n\neq 1$, so the vector $\bar{v}$ is taken to a vector collinear to a certain $\bar{e}_s$ by an operator from Γ. In view of the property of l- and r-sequences specified above, the vectors $\bar{v}_i$ and $\bar{v}_j$ $(i\neq j)$ cannot be taken to vectors collinear to the same $\bar{e}_s$. It follows that we can choose the vectors $\bar{v}_1,\dots,\bar{v}_k$ for $\bar{e}_1,\dots,\bar{e}_k$.

Let φ be an arbitrary affine symmetry of the plane structure ω with linear part a. As we have seen above, for some $\gamma\in\Gamma$, the vector $(\gamma a)\bar{v}_1$ is collinear to a certain $\bar{v}_i$. From the assumptions of the proposition and the properties of l- and r-sequences, it follows that $\bar{v}_i=\bar{v}_1$, that is, $(\gamma a)\bar{v}_1=\lambda\bar{v}_1$. The operator γa takes the development of a saddle link to the development of a saddle link, and since the number of the saddle links parallel to $\bar{v}_1$ is finite, we have $\lambda=\pm1$. But $\gamma a\notin\Gamma$, whenever $a\notin\Gamma$, completing the proof of the first statement.

We proceed to the second statement. Consider an affine symmetry φ with linear part a, $a\bar{v}_1=\pm\bar{v}_1$. Let us prove that $a\in\Gamma$. In view of condition (ii) there is an affine symmetry φ_1 with linear part $a_1\in\Gamma$ such that $\varphi_1\varphi$ takes each saddle link with development $l\bar{v}_1$ into itself, the equation $(a_1a)\bar{v}_1=\pm\bar{v}_1$ being true. Moreover, the symmetry φ_1 can be chosen in such a way that $(a_1a)\bar{v}_1=\bar{v}_1$. Indeed, if $(a_1a)\bar{v}_1=-\bar{v}_1$, then any two bundles of periodic geodesics having a common saddle link with development $l\bar{v}_1$ on the boundary are taken to each other under the affine symmetry $\varphi_1\varphi$. This means that these bundles are congruent. In this case we can "fix up" the symmetry φ_1 using condition (iii).

Now consider an arbitrary bundle of periodic trajectories parallel to $\bar{v}_1$. If there is a saddle link on its boundary invariant under the symmetry $\varphi_1\varphi$, then the same will be true for all the saddle links that bound this bundle on the same side. And if no two saddle links that bound the bundle on the other side are equal in length, then $\varphi_1\varphi$ leaves them invariant as well. In view of condition (iv) and the fact that $\varphi_1\varphi$ leaves fixed the saddle links with development $l\bar{v}_1$, it follows that $\varphi_1\varphi$ leaves fixed all the saddle links parallel to $\bar{v}_1$. In this case the symmetry $\varphi_1\varphi$ takes any bundle of geodesics parallel to $\bar{v}_1$ to itself, so its linear part is a power of the operator b from condition (i), that is, it belongs to Γ. Then a is in Γ as well, completing the proof. $\square$

Below we consider plane structures canonically constructed from rational triangles two of whose angles are of the form π/n and π/m, where n and m are positive integers. The corresponding plane structure and the surface on which it is defined are denoted by $\tilde{\omega}_{n,m}$ and $M_{n,m}$, respectively. The surface $M_{n,m}$ is "pasted" from regular n- and m-gons with equal side length so that each side of each n-gon is identified with a side of a certain m-gon (see [1]). At the same time, all vertices of the regular polygons are identified, forming a singular point of the plane structure $\tilde{\omega}_{n,m}$. Besides this singular point, there are others, the centers of the polygons. The multiplicity of any of these is 1, so they can be thought of as nonsingular points. As a result, we get a new plane structure, which will be denoted by $\omega_{n,m}$. The sides of the regular polygons are the shortest saddle links of the plane structure $\omega_{n,m}$. Cartesian coordinates on the surface $M_{n,m}$ will be chosen in such a way that one of these saddle links be horizontal.

On the face of it, the distinction between the plane structures $\omega_{n,m}$ and $\tilde{\omega}_{n,m}$ is purely formal. However, their stabilizers can differ significantly (see §3).

Let k be an integer, and let l be an integer or ∞, both numbers being no smaller than 2 with at least one of them strictly greater than 2. By $\Gamma_{k,l}$ we shall denote the subgroup in $SL(2,\mathbb{R})$ generated by the elements

$$\sigma_k = \begin{pmatrix} \cos\frac{\pi}{k} & -\sin\frac{\pi}{k} \\ \sin\frac{\pi}{k} & \cos\frac{\pi}{k} \end{pmatrix} \quad \text{and} \quad \tau_{k,l} = \begin{pmatrix} 1 & 2L_{k,l} \\ 0 & 1 \end{pmatrix},$$

where

$$L_{k,l} = \frac{\cos\frac{\pi}{k} + \cos\frac{\pi}{l}}{\sin\frac{\pi}{k}}, \qquad L_{k,\infty} = \frac{\cos\frac{\pi}{k} + 1}{\sin\frac{\pi}{k}} = \operatorname{ctg}\frac{\pi}{2k}.$$

The group $\Gamma_{k,l}$ is a lattice (see [1]). It has one cusp if l is finite and two cusps if $l = \infty$.

Proposition 1.2. *The following equations hold:*

(1) $\Gamma(\omega_{n,n}) = \Gamma(\omega_{2,n}) = \Gamma_{n,2}$ *for an odd* n;
(2) $\Gamma(\omega_{n,n}) = \Gamma(\omega_{2,n}) = \Gamma_{n/2,\infty}$ *for an even* $n \geq 6$;
(3) $\Gamma(\omega_{n,2n}) = \Gamma_{n,3}$ *for any* n.

Proof. The fact that each of the stabilizers in question contains the corresponding lattice was proved in [1, Theorem 4.4]. To prove that the groups coincide, we shall use Proposition 1.1. In all the cases the horizontal direction will be taken as $\overline{v}_1$. The lattice $\Gamma_{n/2,\infty}$ has two cusps; therefore, for the plane structures $\omega_{n,n}$ and $\omega_{2,n}$ (with an even n) we shall choose the direction of $\overline{v}_2$ to make an angle of π/n with the horizontal direction. Then for the plane structures of the form $\omega_{2,n}$, where n is divisible by 4, all the terms in the r-sequence corresponding to the direction $\overline{v}_2$ are equal, and in the r-sequence corresponding to the direction $\overline{v}_1$ one of the terms is half of any of the other if $n \geq 8$. As for the plane structures $\omega_{n,n}$ (with an even n) and $\omega_{2,n}$ (with an even n not divisible by 4), their r-sequences corresponding to the directions $\overline{v}_1$ and $\overline{v}_2$ are of different length (to be exact, the first sequence is one term shorter than the second). Thus we have checked one of the conditions of Proposition 1.1.

Now let us check conditions (i)- (iv). The validity of condition (i) for all the cases in question was established in [1], namely, it is shown there that one of the generators of the corresponding lattice $\Gamma = \Gamma_{n,m}$, the operator $\tau_{n,m}$, can be taken as the operator b. Further, let l be equal to the length of the shortest saddle link. Then $\omega_{2,n}$ (for any n) has one saddle link with the development $l\overline{v}_1$, and $\omega_{n,n}$ (for an even n), as well as $\omega_{n,2n}$, has two saddle links taken one into the other by an affine symmetry with linear part ± 1. Notice that $-1 \in \Gamma_{n,m}$. The saddle links of the plane structure $\omega_{n,2n}$ with development $l\overline{v}_1$ bound two noncongruent bundles, those of all the other plane structures are taken into themselves by an affine symmetry with linear part -1. Finally, only the plane structure $\omega_{n,n}$ with n divisible by 4 has a periodic bundle parallel to $\overline{v}_1$ whose boundary component contains two saddle links of the same length. This is the bundle containing the centers of the regular n-gons that form the surface $M_{n,n}$; after it is deleted, the surface (without the singular point) falls into two connected components each of which contains a saddle link with development $l\overline{v}_1$. Thus conditions (i)–(iv) are verified. □

§2. Symmetric periodic trajectories

As was noticed by Stepin (see, for instance, [5]), a billiard trajectory in a rational polygon starting perpendicularly to a side either is periodic or is a generalized diagonal. The argument can be carried over from billiard trajectories to trajectories (that is, geodesics) of plane structures.

Definition 2.1. An *affine reflection* of a plane structure is an affine symmetry whose linear part is an operator of reflection in a line (not necessarily orthogonal) and which has at least one nonsingular fixed point.

Definition 2.2. A trajectory of a plane structure is said to be *symmetric* if there exists an affine reflection that takes a vector parallel to this trajectory into the opposite one and leaves invariant at least one point of the trajectory.

Proposition 2.1. *A symmetric trajectory of a plane structure is either periodic or a saddle link.*

Proof. Let L be a symmetric trajectory with direction vector $\overline{v}$, and let a point $x \in L$ and affine reflection S be such that $Sx = x$ and $S\overline{v} = -\overline{v}$. Further, suppose that $\overline{u}$ is a vector taken by the reflection S into itself and L_1 is the trajectory passing through x parallel to the vector $\overline{u}$. Each point $y \in L_1$ is invariant under the reflection S (by the way, this implies that the closure of L_1 has no interior points, that is, L_1 is finite). The trajectory L meets L_1 at the point x. If L is not a saddle link, it will meet L_1 once again, say, at a certain point y. Denote by $[x, y]$ the segment of the trajectory L from the point x to the point y. Since $S\overline{v} = -\overline{v}$, the segments $S([x, y])$ and $[x, y]$ are adjacent segments of L, the entire trajectory L being the union of these segments. It follows that the trajectory L is periodic. □

The assertion about billiard trajectories starting perpendicularly to a side of the polygon turns out to be a particular case of the one proved above. Indeed, let x be a point on a side a of a rational polygon Q, and let L be the billiard trajectory starting from x perpendicularly to the side a. Consider a surface M with plane structure into which the polygon Q can be embedded. The billiard trajectory L can be straightened into a trajectory L_1 of the plane structure. The reflection in the side a generates an affine (orthogonal) reflection S of the plane structure. The trajectory L_1 is obviously symmetric under the reflection S, so by the proposition proved above it is either periodic or a saddle link. It follows that the billiard trajectory L either is periodic or is a generalized diagonal.

In §1 we considered plane structures whose stabilizer is a lattice. In this case Proposition 2.1 can be conversed.

Proposition 2.2. *For plane structures of the form $\omega_{n,n}$, $\omega_{2,n}$, or $\omega_{n,2n}$, each periodic trajectory and each saddle link is symmetric.*

Proof. Suppose that S is an affine reflection and T is an affine symmetry of a plane structure, and let L be a trajectory symmetric under the reflection S. Then the trajectory $T(L)$ is symmetric under the affine reflection TST^{-1}. Thus an affine symmetry takes symmetric trajectories into symmetric trajectories. However, it follows from the proofs of Propositions 1.1 and 1.2 that any finite trajectory (a periodic one or a saddle link) of any plane structure of the form indicated in the statement is transformed by an affine symmetry either into a trajectory parallel to the shortest saddle link or, for the plane structures $\omega_{n,n}$ and $\omega_{2,n}$ with an even n, into a trajectory parallel to or making an angle of π/n with the shortest saddle link. But all these trajectories are symmetric and the corresponding affine reflection can be chosen to be orthogonal. □

§3. Bundles of periodic billiard trajectories

Let Q be a rational polygon, and let ω be the plane structure corresponding to Q on the surface M. In [1] we considered the values $N_0(R)$, $N(R)$, $S(R)$, $N^*(R)$, and $S^*(R)$ denoting the number of saddle links of ω of length not greater than R, the number and total area of "one-run" bundles of periodic trajectories of length not greater than R, and the same number and total area again, but with allowance made for multiple bundles (trajectories in a multiple bundle are trajectories of smaller length covered several times), respectively. By analogy, let us introduce the values $\widetilde{N}_0(R)$, $\widetilde{N}(R)$, $\widetilde{S}(R)$, $\widetilde{N}^*(R)$, and $\widetilde{S}^*(R)$ characterizing a billiard flow in Q. Here generalized diagonals are counted instead of saddle links, and bundles of periodic billiard trajectories instead of bundles of periodic geodesics. By the area of a one-run bundle of periodic trajectories here we understand the product of its length by its width (it is greater than the area of the subset of Q occupied by the trajectories of the flow). The area of a multiple bundle is, by definition, the area of the corresponding one-run bundle.

Denote by k the order of the group K generated by the linear parts of reflections in the sides of the polygon Q. The natural projection $\pi\colon M \to Q$ is k-sheeted; therefore, a generalized diagonal of the polygon, as a rule, is the image of k saddle links of the plane structure ω. Similarly, a bundle of periodic billiard trajectories is the image of k periodic bundles of trajectories of the geodesic flow on M (of the same area). The exception is the generalized diagonals and periodic trajectories whose segments are oriented along the fixed vectors of the reflections from K. These generalized diagonals and periodic bundles (finite in number) can have $k/2$ or k preimages. (Notice that a periodic bundle with $k/2$ preimages contains one trajectory half as long as the bundle; it is a periodic trajectory with an odd number of segments [5].)

Thus we have

$$N_0(R) = k\widetilde{N}_0(R), \qquad N(R) = k\widetilde{N}(R),$$

and so on up to an additive constant. If the stabilizer $\Gamma(\omega)$ is a lattice, then, by [1, Theorem 3.13], all the values

$$\widetilde{N}_0(R), \quad \widetilde{N}(R), \quad \widetilde{S}(R), \quad \widetilde{N}^*(R), \quad \widetilde{S}^*(R)$$

are asymptotically of the form $cR^2 + o(R^2)$ with a positive constant c as $R \to \infty$. If the stabilizer is a lattice for a plane structure that differs from ω only by removable singularities, then, as in the previous case, the values $\widetilde{S}(R)$ and $\widetilde{S}^*(R)$ will have the same asymptotics (because the values $S(R)$ and $S^*(R)$ corresponding to the plane structure do not depend on removable singularities); however, this is not necessarily true for the other values.

Proposition 3.1. *The stabilizers of the plane structures $\widetilde{\omega}_{2,n}$ and $\widetilde{\omega}_{n,n}$ are lattices for any even $n > 2$ and are not lattices for any odd $n > 3$.*

Proof. Suppose that n is even. Let us show that the lattice $\Gamma_{n/2,\infty}$, the stabilizer of the plane structures $\omega_{2,n}$ and $\omega_{n,n}$, is the stabilizer for $\widetilde{\omega}_{2,n}$ and $\widetilde{\omega}_{n,n}$, as well. The fact that the plane structures $\widetilde{\omega}_{2,n}$ and $\widetilde{\omega}_{n,n}$ allow for an affine symmetry with linear part $\sigma_{n/2}$ follows from their construction. Further, let φ be an affine symmetry of the plane structure $\omega_{2,n}$ (or $\omega_{n,n}$) with linear part $\tau_{n/2,\infty}$ that leaves invariant horizontal saddle links. It will suffice to show that it is also an affine symmetry of the structure $\widetilde{\omega}_{2,n}$ ($\widetilde{\omega}_{n,n}$, respectively), that is, takes the singular points of the latter one to another. The plane structure $\widetilde{\omega}_{n,n}$ differs from $\omega_{n,n}$ in two singular points, the centers of the n-gons constituting the surface $M_{n,n}$. If n is not divisible by 4, they lie on the horizontal saddle links of $\omega_{n,n}$ and φ leaves them fixed. If n is divisible by 4, they belong to the trajectory that divides in half one of the horizontal bundles of trajectories of $\omega_{n,n}$; the distance between the points is half the length of the bundle, so they are swapped by φ. Exactly the same way, the singular points of $\widetilde{\omega}_{2,n}$ not on the horizontal saddle links of the plane structure $\omega_{2,n}$ are located on the trajectories that divide in half the bundles of horizontal trajectories of $\omega_{2,n}$, two points on a trajectory; the distance between the points in such a pair is half the length of the trajectory and the symmetry φ swaps the points in each pair.

If n is odd, the singular points of $\widetilde{\omega}_{n,n}$, which are the centers of the regular n-gons constituting $M_{n,n}$, split the horizontal bundle of trajectories of the plane structure $\omega_{n,n} = \omega_{2,n}$ to which they belong into three parts of widths

$$d\sin(\pi/(2n)), \quad d\bigl(\sin(3\pi/(2n) - \sin(\pi/(2n))\bigr), qd\sin(\pi/(2n)),$$

where d is the distance from the center of the regular n-gon to its vertex. The ratio of the first two of these values is equal to

$$\frac{\sin\frac{3\pi}{2n} - \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} = \frac{2\sin\frac{\pi}{2n}\cos\frac{\pi}{n}}{\sin\frac{\pi}{2n}} = 2\cos\frac{\pi}{n},$$

which is an irrational number for $n > 3$. The singular points of the plane structure $\widetilde{\omega}_{2,n}$ distinct from the singular points of $\widetilde{\omega}_{n,n}$ divide in half all the bundles of horizontal trajectories of the plane structure $\omega_{n,n}$, so for $\widetilde{\omega}_{2,n}$, as well as for $\widetilde{\omega}_{n,n}$, the length-to-width ratios for horizontal bundles are not pairwise commensurable. Thus the Veech alternative does not hold and the stabilizers $\Gamma(\widetilde{\omega}_{2,n})$ and $\Gamma(\widetilde{\omega}_{n,n})$ are not lattices. □

Corollary 3.1. *For a right triangle with an acute angle of π/n and an isosceles triangle with base angles of π/n, where n is an even number, the values $\widetilde{N}_0(R)$, $\widetilde{N}(R)$, $\widetilde{S}(R)$, $\widetilde{N}^*(R)$, and $\widetilde{S}^*(R)$ have asymptotics of the form $cR^2 + o(R^2)$, where c is a positive constant, as $R \to \infty$.*

§4. *d*-unstable billiard trajectories

In connection with bifurcations of periodic trajectories of billiards in polygons, Stepin introduced the notion of deformational stability (d-stability) of these trajectories. In [5] a number of statements on deformational stability or instability of periodic billiard trajectories in various polygons were proved. Below we prove another result along this line.

Definition 4.1. A periodic billiard trajectory in a polygon Q is called *d-stable* if for any $\varepsilon > 0$ in each polygon obtained from Q by a small enough variation of its sides and angles, there is a periodic billiard trajectory with the same number of segments entirely contained within the ε-neighborhood of the initial trajectory.

Lemma 4.1 [5]. *Let $a_1, \dots, a_l$ be the sides of the polygon Q and let $W = a_{i_1} \cdots a_{i_m}$ be a cyclic word associated with a periodic billiard trajectory (that is, the succession of sides visited by a point moving along the trajectory). If m is even, then this trajectory is d-stable if and only if the alternating sum $+a_{i_1} - a_{i_2} + \cdots - a_{i_m}$ is equal to 0 (as an element of the free $\mathbb{Z}$-module with the basis $a_1, \dots, a_l$).*

Lemma 4.2. *Suppose that the measures of the angles at two vertices joined by the generalized diagonal bounding a bundle of periodic billiard trajectories in a polygon are of the form $\pi/(2n)$, $n \in \mathbb{N}$. Then all the trajectories in this bundle are d-unstable.*

Proof. Label the polygon's sides arbitrarily. Let L be the generalized diagonal in question; denote by A and B its endpoints and suppose that the sides visited by a point running along L from A to B are, in succession, $x_1, x_2, \dots, x_m$. Let the angles of the polygon at the vertices A and B be $\pi/(2n)$ and $\pi/(2l)$, respectively. A periodic billiard trajectory from the bundle bounded by the generalized diagonal L can be described as follows. The billiard ball, starting its motion near the vertex A, moves in parallel to the generalized diagonal L to its end, rebounds $2l$ times alternately in the sides c and d issuing from the vertex B of the polygon, then returns moving in parallel to L again, but on the other side of the diagonal, and, at the end of its route, rebounds $2n$ times in the sides a and b issuing from the vertex A to end up at its starting position. Thus the word associated with the periodic trajectory is

$$W = x_1 x_2 \cdots x_m (cd)^l x_m \cdots x_2 x_1 (ab)^n$$

(the ith power of a subword in this notation stands for its i-fold iteration). The alternating sum of the letters in the word W equals

$$S = (-1)^m l(c - d) + n(a - b).$$

If the vertices A and B do not coincide, this sum is not zero, because in this case at least three of the sides a, b, c, and d are distinct from one another. The case $A = B$ requires a more detailed treatment. In this case, the first segment of the periodic trajectory and its $(m+1)$st segment, which are parallel to the first and $(m+1)$st (that is, the last) segments of L, respectively, are located to the left or to the right of the corresponding segments of L (the left and right sides are determined by the initial orientation of L). More exactly, both segments of the trajectory are on the same side (both on the left or both on the right) of the segments of the generalized diagonal if m is even, and on different sides if m is odd. Since the first segment of the billiard trajectory starts on the side b and its $(m+1)$st segment ends on the side c, the first segment of L issues from the vertex A and its $(m+1)$st segment enters this vertex, we see that either $a = c$ and $b = d$ (for an even m) or $a = d$ and $b = c$ (for an odd m). In any case,

$$(-1)^m (c - d) = a - b, \qquad S = 2n(a - b) \neq 0.$$

Thus $S \neq 0$, and so, according to Lemma 4.1, the periodic billiard trajectory in the bundle bounded by the generalized diagonal L is d-unstable. □

Theorem 4.1. *In a right triangle with an acute angle of $\pi/(2n)$, $n \in \mathbb{N}$, all periodic billiard trajectories are d-unstable.*

Proof. Recall that the surface with plane structure corresponding to a right triangle with an acute angle of $\pi/(2n)$, $n \in \mathbb{N}$, is obtained from the regular $2n$-gon by identification of its opposite sides. Singular points of the plane structure correspond to the vertices of the triangle. In particular, the vertices with the angles $\pi/2$ and $\pi/(2n)$ turn into $n+1$ singular points of multiplicity 1 (the center of the $2n$-gon and the midpoints of its sides), and the third vertex of the triangle turns into one more singular point p, whose multiplicity is greater than 1 for $n > 3$. It follows from Propositions 1.2 and 3.1 (and their proofs) that any bundle of periodic geodesics of the plane structure in question can be transformed under a certain affine symmetry φ into a bundle parallel to a side or a diagonal of the $2n$-gon. Each of these bundles, as is easy to see, has a saddle link (on the boundary) none of whose endpoints coincides with p. Now we notice that for $n > 3$ the symmetry φ takes the point p to itself (because in this case p is the only singular point with multiplicity greater than 1), and for $n = 2$ or 3 it can be chosen in such a way as to leave the point p fixed. Therefore, any bundle of periodic geodesics has a similar saddle link on the boundary. It follows that each bundle of periodic billiard trajectories in the initial triangle has a generalized diagonal on the boundary both of whose endpoints are at the vertices of the angles measuring $\pi/2$ or $\pi/(2n)$. By Lemma 4.2, this completes the proof. □

In [5] a different method was used to prove the absence of d-stable periodic billiard trajectories in three polygons: in a rectangle and in the right triangles with an acute angle of $\pi/4$ or $\pi/6$. We see that for the first two of these polygons this statement follows from Lemma 4.2; besides, for the triangles it turns out to be a particular case of Theorem 4.1.

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