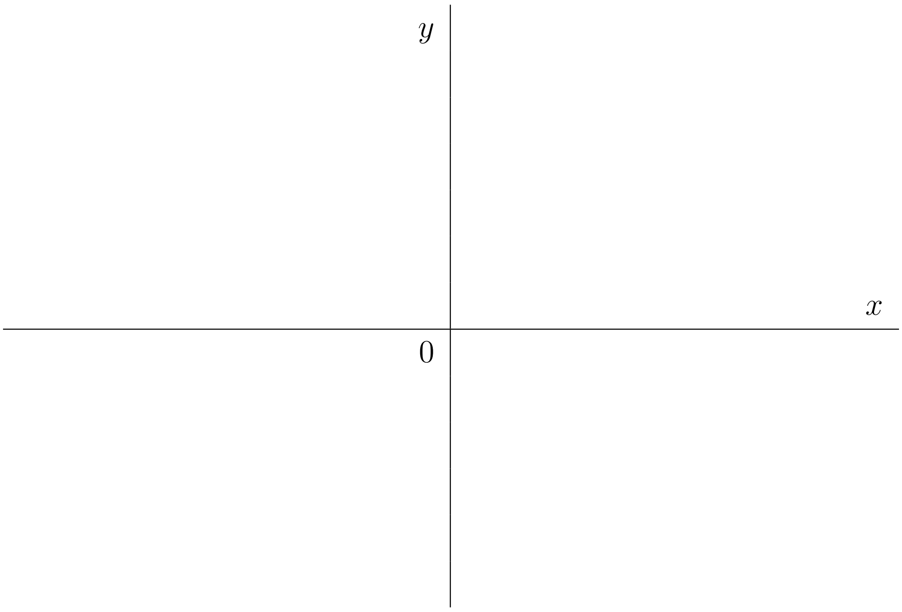
Absolute Extremum: tutorial exercises

Example 1. *Find the absolute minimum and absolute maximum of* $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ *on the region* $D = \{(x,y) | -1 \le x \le 1, -1 \le y \le 1\}$.

Sketch the region D here:



Solution. *STEP 1.* Find all the critical points that lie inside the given rectangle. To do this you need the two first order derivatives. Solve the following system

$$f_x = \boxed{\qquad\qquad} = 0$$

$$f_y = \boxed{\qquad\qquad} = 0$$

To solve the system you can solve the second equation for y to get, $y = \boxed{\qquad\qquad}$. Plugging this into the first equation will give you

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad} = 0.$$

Then you must have $x = 0$ or $x = \pm\boxed{\quad}$. Recall that we only want critical points in the region that were given. That means that we only want critical points for which $-1 \le x \le 1$. The only value of x that will satisfy this is the first one so we can ignore the last two for this problem.[1]

You aren't going to be classifying the critical points. Thus, you don't need the Second Order Derivatives. Plugging $x = 0$ into the equation for y gives , $y = 0$. So, the single critical point in the

[1]Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

region D (and again, that's important), is (☐ , ☐). You need to get the value of the function at that critical point: $f(0,0) =$ ☐ . You will compare this to values of the function found in the next step and take the largest and the smallest as the absolute extrema of the function in D.

Step 2. Now you need to find the absolute extrema of the function along the boundary of D. First get the values of f at corner points:

☐

The boundary of D is given by the following conditions:

- right side: $x = 1$, $-1 \le y \le 1$
- left side: $x =$ ☐ , $-1 \le y \le 1$
- upper side: $y = 1$, ☐
- lower side: ☐ , ☐

Along the right side you know that $x = 1$. Using this, define a new function as follows,

$$g(y) = f(1, y) = \square$$

Find the critical points of $g(y)$ in the range $-1 < y < 1$ and then evaluate $g(y)$ at the critical points:

$$g'(y) = 8y - 2 = 0 \quad \Rightarrow \quad y = 1/4.$$

This is in the range and so you will need to evaluate $g(\frac{1}{4}) = 4.75$. Notice that, using the definition of $g(y)$ this is also function value for $f(x, y)$:

$$g(\frac{1}{4}) = f(1, \frac{1}{4}) = 4.75$$

.

Now do the left side of D. Notice that, for this boundary, this is the same function as you looked at for the right side: $f(1, y) = f(-1, y)$.[2] So, we will have: $f(-1, \frac{1}{4}) = 4.75$

Look at the upper side. Define a new function:

$$h(x) = f(x, 1) = \square .$$

Find the critical points of $h(x)$ in the range $-1 < x < 1$ and then evaluate $h(x)$ at the critical points:

[2] This will not always happen.

Look at the lower side (Again, define a new function and find its critical point/s in the range $-1 < x < 1$.)

Step 3. Collect up all the function values for that youve computed in this problem:

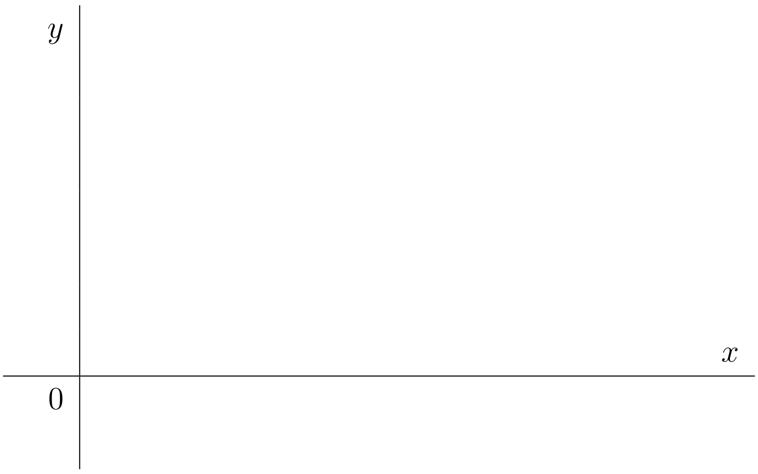
(x, y)	$(0, 0)$	$(1, -1)$	$(1, 1)$	$(1, \frac{1}{4})$	$(-1, 1)$	$(-1, -1)$	$(-1, \frac{1}{4})$	$(0, 1)$	$(0, -1)$
$f(x, y)$	4	11	7	4.75	7	11	4.75	8	8

The absolute minimum is at $(0, 0)$ and its value is 4 and the absolute maximum occurs at $(\pm 1, -1)$ and its value is 11.

Example 2 *Find the absolute minimum and absolute maximum of* $f(x, y) = 2x^2 + y^2 + y + 1$ *on the region* $D = \{(x, y) | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$.

Solution.

Sketch the region D here:



STEP 1. Find all the critical points that lie inside D. To do this you need the two first order derivatives in order to solve the system

$$f_x = \boxed{} = 0$$

$$f_y = \boxed{} = 0$$

Write its solution here: $\boxed{}$. Recall that we only want critical points in the region that were given.

You arent going to be classifying the critical points. Thus, you dont need the Second Order Derivatives. Evaluate the value of f at a critical point only if it belongs to D.

Step 2. Now you need to find the absolute extrema of the function along the boundary of D.

First get the value of f at corner points:

$$\boxed{}$$

Describe/parametrize the boundary of D.

- right side: $x = 0$, $0 \le y \le 1$
- lower side: $y = 0$, $0 \le x \le 1$
- left side (arc of the circle): $x = \sqrt{1 - y^2}$, $0 \le y \le 1$. (Notice here that there are many other parametrizations that will also work here. For example, you can use $x = \cos t, y = \sin t$ for $0 \le t \le \pi/2$.)

Along the right side you know that $x = 0$. Using this, define a new function as follows,

$$g(y) = f(0, y) = \boxed{}$$

Find the critical points of $g(y)$ in the range $0 < y < 1$ and then evaluate $g(y)$ at the critical point/s:

$\boxed{}$ Now do the lower side of D. Define a new function:

$$h(x) = f(x, 0) = \boxed{}.$$

Find the critical points of $h(x)$ in the range $0 < x < 1$ and then evaluate $h(x)$ at the critical point/s:

$\boxed{}$

Look at the left side. Again, define a new function

$$G(y) = f(\sqrt{1-y^2}, y) = \boxed{} = -y^2 + y + 3$$

and find its critical point/s in the range $0 < y < 1$ from the condition $G'(y) = 0$.

$$\boxed{}$$

Step 3. Collect up all the function values for that you've computed in this problem:

(x, y)	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$f(x, y)$	1			13/4

The absolute minimum is at $(0, 0)$ and its value is 1 and the absolute maximum occurs at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and its value is $13/4$.