MATH 251 Extra Practice (Final Exam)

1. Verify Stokes' Theorem $\int\int_S \text{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{r}$ for the vector field $\vec{F} = \langle yz^2, -xz^2, z^3\rangle$ and the cylinder $x^2+y^2=9$ for $1 \le z \le 2$ oriented out.

2. Find the absolute maximum and minimum values of the function $f(x,y) = 1+xy-x-y$ on the region D bounded by the parabola $y=x^2$ and the line $y=4$.

3. Evaluate the surface integral $\int\int_S y\,dS$, where S is the part of the plane $3x+2y+z=6$ that lies in the first octant.

4. Find the mass of the conical surface $z=\sqrt{x^2+y^2}$ for $z \le 2$ with density $\rho(x,y) = x^2+y^2$.

5. Verify Green's Theorem for the vector field $\mathbf{F}(x,y) = \langle -x^2y, xy^2\rangle$ on the region inside the circle $x^2+y^2=16$.

6. Find the absolute maximum and minimum of the function $f(x,y) = \frac{1}{8}x^4 - x^2 + 2 + y^2$ on the square determined by the four points $A(4,4)$, $B(-4,4)$, $C(4,-4)$, $D(-4,-4)$.

7. Find the absolute maximum and minimum of the function $f(x,y) = x^3 - xy + y^2 - x$ on $D = \{(x,y) : x \ge 0, y \ge 0, x+y \le 2\}$.

8. Determine the equation of the plane tangent to the surface $\vec{r}(u,v) = \langle u^2+v, u^2-v, 2uv\rangle$ at the point $(10,8,6)$.

9. Let $\vec{F}(x,y) = \langle x+y^2, 2xy+y^2\rangle$.

   a) Show that $\vec{F}$ is conservative vector field.

   b) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where C is any path from (-1,0) to (2,2).

10. Let $\vec{F}(x,y,z) = \langle 4xe^z, \cos y, 2x^2e^z\rangle$.

    a) Show that $\vec{F}$ is conservative vector field.

    b) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where $C : \mathbf{r}(t) = \langle t, t^2, t^4\rangle$, $0 \le t \le 1$.

11. Consider the surface S parameterized by $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z\rangle$ where $0 \le \theta \le \pi, \quad 0 \le z \le 2$.

    a) Find the normal vector $\vec{N}(\theta,z)$ to the surface S;

    b) Find $|\vec{N}(\theta,z)|$;

    c) Find $\displaystyle\int\int_S yz\,\text{dS}$.

12. Apply the Divergence Theorem to compute $\displaystyle\int\int_S \vec{F}\cdot d\vec{S}$ for the vector field $\vec{F}(x,y,z) = \langle x^3, y^3, x+y\rangle$ over the complete boundary S of the solid paraboloid $\{(x,y,z) : x^2+y^2 \le z \le 1\}$ with outward normal.

13. Verify that Stokes' Theorem is true for the vector field $\mathbf{F} = \langle x^2, y^2, z^2\rangle$ where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, and S has upward orientation.

14. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, yz, zx\rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counterclockwise as viewed from above.

15. Evaluate $\iint_S y\, dS$ where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.
(*Hint:* parametrize S, find the parameter domain, normal and its magnitude.)

16. Apply the Divergence Theorem to compute
$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$
for the vector field
$$\mathbf{F}(x, y, z) = \langle x^3 + \sin(yz), y^3, y + z^3\rangle$$
over the complete boundary S of the solid hemisphere
$$\{(x, y, z) : x^2 + y^2 + z^2 \le 1, z \ge 0\}$$
with outward normal.

17. Verify Stokes' Theorem
$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$$
for the vector field $\mathbf{F} = \langle y, -x, x^2 + y^2\rangle$ and the surface S which is the part of the paraboloid $z = x^2 + y^2$ between $z = 0$ and $z = 1$ with normal pointing up and in.
Use the following steps:
    (a) Sketch the surface S, indicate its boundary, indicate and explain the orientations (on S and on ∂S).
    (b) Compute the line integral.
    (*Hint:* parameterize the boundary circle ∂S and compute $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$.)
    (c) Compute the surface integral.
    (*Hint:* parametrize the paraboloid, indicate the parameter domain, find the normal and determine its direction, compute curl$\mathbf{F}$ and $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$).

18. Find the point where the line $x = 2-t, y = 1+3t, z = 4t$ intersects the plane $2x-y+z = 4$. At this point $x + y + z =$

a. 9

b. -3

c. 0

d. 3

e. -9

19. The equation of the plane tangent to the surface $ze^{\frac{xz}{y}} = 1$ at $(x, y, z) = (0, \frac{1}{2}, 1)$ is

a. $\frac{1}{2}x + z - 1 = 0$

b. $x = 0$

c. $2x + z - 1 = 0$

d. $2x + y + z = \frac{3}{2}$

e. $2x - y + z = \frac{1}{2}$

20. Use the linear approximation to $f(x, y) = \sqrt{x}e^y$ to estimate (approximate) the value $\sqrt{0.99}e^{0.02}$

a. 1.03

b. 0.975

c. 1.025

d. 1.015

e. 1.01

21. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle x, e^y, 2z \rangle$ and C is any path from (5,1,2) to (-1,1,3).

Hint: Find a potential.

a. 7

b. 25

c. -25

d. 0

e. -7

22. The temperature at a point (x, y, z) is given by $T(x, y, z) = e^{-(x+y+z)^3}$. The maximum rate of change of T at the point $P(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ is

a. $\sqrt{3}$

b. e^{-1}

c. $-3\sqrt{3}e^{-1}$

d. $\sqrt{3}e^{-1}$

e. $3\sqrt{3}e^{-1}$

23. For the function $f(x, y) = xy^2 + x^3 - 2xy$ the point $(x, y) = (\frac{1}{\sqrt{3}}, 1)$ is

a. a local minimum

b. a local maximum

c. a saddle point

d. not a critical point

e. is a critical point but the Second Derivative Test fails.

24. A particle moves along the curve $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ from the point $(0, 1, 0)$ to the point $(0, -1, \pi)$ due to the force $\vec{F}(x, y, z) = \langle y, -x, \frac{2z}{\pi} \rangle$. The work done by the force is

a. $1 + \frac{1}{\pi}$

b. 3π

c. π

d. $\frac{1}{\pi}$

e. 2π

25. The integral

$$\int_C (x^2y + \frac{y^2}{2})\,\mathrm{d}x + (xy + \frac{x^3}{3} + 3x)\,\mathrm{d}y$$

where C is the positively oriented triangle with vertices $(0, 0)$, $(1, 0)$,$(0, 1)$.

Hint: Use Green's Theorem.

a. $-\frac{3}{2}$

b. $-\frac{1}{12}$

c. $\frac{3}{2}$

d. 0

e. $\frac{1}{12}$

26. Compute $\iint_S \vec{F} \cdot \mathrm{d}\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x^3 + z \sin y, y^3 + x^2y, x^2z + 2y^2z \rangle$$

over the complete boundary S of the solid cylinder

$$\{(x, y, z) : x^2 + y^2 \leq 4, 0 \leq z \leq 5\}$$

with outward normal.

Hint: Use the Divergence Theorem.

a. $\frac{400}{3}\pi$

b. $\frac{80}{3}\pi$

c. 100π

d. 200π

e. 40π

27. Let D be the region bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant. Find the mass of the lamina that occupies the region D if the density function is $\rho(x, y) = y$.

a. $\frac{1}{2}$

b. 1

c. $\frac{1}{12}$

d. $\frac{2}{3}$

e. $\frac{1}{4}$

28. Find the volume of the solid region which lies above the paraboloid $z = x^2 + y^2$ and below the cone $z = \sqrt{x^2 + y^2}$.

a. $\frac{\pi}{2}$

b. $\frac{\pi}{6}$

c. $\frac{\pi}{3}$

d. π

e. 2π