11.2: Vectors and the Dot Product in Three Dimensions

REVIEW

DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0,0,0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.

Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:

$$\mathbf{a} || \mathbf{b} \quad \Leftrightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector`: $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector`: If $\mathbf{a} \neq \mathbf{0}$, then $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

Standard Basis Vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$ and

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Dot Product of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is a NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If θ is the *angle* between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

DEFINITION 2. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

and

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

DEFINITION 3. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

11.3: Cross product

Determinant of 2×2 and 3×3 matrices.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

or copy the first two columns onto the end and then multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

- `THE CROSS PRODUCT IN COMPONENT FORM:`

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

REMARK 4. The cross product requires both of the vectors to be three dimensional vectors.

REMARK 5. The result of a dot product is a number and the result of a cross product is a VECTOR!!!

To remember the cross product component formula use the fact that the cross product can be represented as the determinant of order 3:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties:

$\mathbf{a} \times \mathbf{a} = 0$

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

$(\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b}) = \alpha(\mathbf{a} \times \mathbf{b}), \quad \alpha \in \mathbb{R}$

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

• GEOMETRIC INTERPRETATION OF THE CROSS PRODUCT:

Let θ be the angle between the two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$. Then

1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin\theta =$*the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$*;*

2. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$;

3. the direction of $\mathbf{a} \times \mathbf{b}$ is determined by "right hand" rule: *if the fingers of your right hand curl through the angle* θ *from* $\mathbf{a}$ *to* $\mathbf{b}$*, then your thumb points in the direction of* $\mathbf{a} \times \mathbf{b}$*.*

FACT: $\quad \mathbf{a} \| \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \times \mathbf{b} = 0.$

• **SCALAR TRIPLE PRODUCT** of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Note that the scalar triple product is a NUMBER.

FACTS:

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

2. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3. $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| =$ *the volume of the parallelepiped determined by* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$*.*