Introduction

1. Ordinary differential equation (ODE) is an equation for one unknown function of one variable that involves the derivatives of this function. We are generally trying to solve these for the unknown function.

   If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter is called an **independent variable**.

2. The order of ODE is the order of the highest derivative in the equation.

3. State the order of the DE $y'' + y = 0$, and confirm that the functions in the family $y(t) = C_1 \sin t + C_2 \cos t$ are solutions.

1. Solutions of some differential equations: first order linear equations with constant coefficients (section 1.2)

1. If the differential equation can be rearranged to only have linear terms in y and its derivatives then it is a **linear** differential equation. If an ODE is not linear, then we call it **nonlinear**.

2. (a) Find $y(t)$ satisfying the equation $y' = 0$.
   (b) Find $y(t)$ if $y' = 0$ and $y(t_0) = y_0$.

3. Typically we need to find the **particular solution** that satisfies a condition of the form $y(t_0) = y_0$. The latter is called an **initial condition** and the problem of finding a solution of ODE that satisfies the initial condition is called an **initial value problem (IVP).**

4. The graph of a solution of ODE is called an **integral curve** for the equation. So, the general solution produces a family of integral curves corresponding to choice for the arbitrary constants.

5. Geometrically, the initial condition $y(t_0) = y_0$ has the effect of isolating the integral curve that passes through the point (t_0, y_0) in the (t, y) plane from the complete family of integral curves.

6. (a) Solve ODE: $y' = t$ and sketch some typical integral curves.
   (b) Find the solution satisfying initial conditions $y(1) = \frac{3}{2}$.
   (c) Solve IVP: $y' = t$, $y(t_0) = y_0$.

7. Suppose that $f(t)$ is a given continuous function. Solve ODE: $y'(t) = f(t)$.

8. (a) Solve ODE: $y' = y$.
   (b) Solve IVP: $y' = y$, $y(t_0) = y_0$.

9. Solve $y' = ay$, where a is an arbitrary real constant.

10. Consider $y' = ay + b$, where a and b are arbitrary real constants.
    (a) Find general solution.

(b) Find solution satisfying the initial condition $y(0) = y_0$.

11. Solution that doesn't change with time (i.e. $y(t) = y_0$ for any t pr shortly $y(t) \equiv y_0$) is called an **equilibrium solution**. The corresponding y_0 is called an **equilibrium or stationary point.**

REMARK 1. *In general the equation of the form* $\frac{dy}{dt} = f(y)$ *(at least as* $f(y) \neq 0$*) is equivalent to the equation* $\frac{dt}{dy} = \frac{1}{f(y)}$*. So by switching the role of the dependent and the independent variables (i.e. swapping* t *and* y*) we arrive here to the same type of equations as in the item 7 above.*

12. **A model of free-fall motion retarded by air resistance.**

Suppose that an object with mass m falls through air toward Earth. Assume that the only forces acting on the object are gravity and air resistance (=drag force). Denote by γ the drag coefficient.

(a) Formulate an ODE that describes the motion.

(b) Find general solution of the ODE obtained in the previous item.

(c) Find a particular solution of the ODE obtained in (a) satisfying the initial condition $v(0) = v_0$.

2. Separable Equations (section 2.2)

1. The general first order ODE (solved with respect to the first derivative) has the form

$$y' = F(t, y). \tag{1}$$

Most of such equations cannot be solved explicitly. We can only claim the existence and the uniqueness of solutions with given initial conditions under some quite general assumptions on the function F in the right-hand side of (1)(these conditions will be discussed later in mode detail).

There are few classes of first order ODE which can be solved equation. The first type of such ODEs that we will study is called *separable equations* and it generalizes all equations we solved so far.

Separable ODE is an ODE that is expressible in the forms:

$$y' = f(t)g(y).$$

In other words, the function $F(t, y)$ of two veriables in the right hand side of (1) can be expressed as a product of two functions of single variables: one of t and one of y. Obviously, not any (and in fact very few) functions of two variables can be expressed as a product of two functions of single variables.

2. Steps to solve a separable ODE:

(a) More rigorous way: Divide the equation by $f(y)$ (assuming that $f(y_0) \neq 0$), the case when $f(y_0) = 0$ has to be considered separately) obtaining
Less rigorous but formally valid way that you may use in your solutions We can say that we separate variables, i.e. rewrite the given equation in the differential form:

$$\frac{1}{f(y)} \mathrm{d}y = g(t)\mathrm{d}t. \tag{2}$$

(b) More rigorous way Integrate both sides of the equation (*) in the handwritten part above with respect to t and make a change of variables in the resulting integral in the left-hand side
Less rigorous wayIntegrate both sides of (2):

$$\int \frac{\mathrm{d}y}{f(y)} = \int g(t)\mathrm{d}t. \tag{3}$$

(c) If $P(y)$ and $G(t)$ are antiderivatives of $\frac{1}{f(y)}$) and $q(t)$ respectively, then the equation $P(y) = G(t) + C$ will define a family of solutions implicitly. (Note, in some cases it may be possible to solve this equation for y to find the solution explicitly.)

3. (a) Find general solution of $ty' = -y \quad (t > 0)$ and sketch some typical integral curves.
   (b) Solve IVP: $ty' = -y \quad (t > 0), \quad y(4) = 2$.

4. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\ln x}{xy + xy^3}, \qquad x > 0.$

5. Separate variables in the following equations:

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{5\sin x - y^3}$

   (b) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = 1 + t - u - ut.$