# 23: Systems of homogeneous linear equations: the case of complex eigenvalues (section 7.6)

1. Let $A$ be an $n \times n$ matrix with real entries. It may happen that the characteristic equation $\det(A - \lambda I) = 0$ has a complex root $\lambda$, i.e. $\lambda$ is a complex eigenvalue of $A$. Then a corresponding eigenvector $v$ has complex components, because it satisfy the linear algebraic system of equations $(A - \lambda I)v = 0$ and the matrix $A - \lambda I$ already have complex entries on its diagonal.

2. If $\lambda$ is a complex eigenvalue of the real matrix $A$ with an eigenvector $v$, then $\bar{\lambda}$ (the complex conjugate of $\lambda$) is an eigenvalue of $A$ with an eigenvector $\bar{v}$. In other words complex eigenvalues of a real matrix $A$ come in pair of complex conjugate ones.

3. Hence if $\lambda$ is a complex eigenvalue of a real matrix $A$ with an eigenvector $v$ then both $e^{\lambda t}v$ and $e^{\bar{\lambda}t}\bar{v}$ are solutions of the system $X' = AX$. These solutions are vector valued functions having complex components. Then by the superposition principle, the vector valued functions $\mathrm{Re}(e^{\lambda t}v)$ and $\mathrm{Im}(e^{\lambda t}v)$, i.e. the vector valued function of the real parts of the components of $e^{\lambda t}v$ and the vector valued function of the imaginary parts of the components of $e^{\lambda t}v$ are solution of the same system $X' = AX$:

4. Note that any vector $v$ with complex component can be uniquely represented as $v = a + ib$ where $a$ and $b$ are vectors with real components, the real and imaginary parts of $v$ respectively.

5. If $\lambda = \alpha + i\beta$, $\beta \neq 0$ is an eigenvalue of a real matrix $A$ and $v = \mathbf{a} + i\mathbf{b}$ is an eigenvector of $\lambda$, where $a$ and $b$ are vectors with real components then

   $e^{t\lambda} v =$

   i.e.

$$\boxed{\begin{aligned} \mathrm{Re}(e^{\lambda t} v) &= e^{\alpha t}(\cos(\beta t)\,\mathbf{a} - \sin(\beta t)\,\mathbf{b}), \\ \mathrm{Im}(e^{\lambda t} v) &= e^{\alpha t}(\sin(\beta t)\,\mathbf{a} + \cos(\beta t)\,\mathbf{b}) \end{aligned}}$$

6. Case $n = 2$. If $\lambda = \alpha$ is a complex eigenvalue of the coefficient matrix $A$ in thesystem $X' = AX$ and $v$ is a corresponding eigenvector then

   - $$\left\{ e^{\lambda t} v, e^{\bar{\lambda} t} \bar{v} \right\} \tag{1}$$
     is a fundamental set of (complex) solutions of the system $X' = AX$.
   - $$\left\{ \mathrm{Re}(e^{\lambda t} v), \mathrm{Im}(e^{\lambda t} v) \right\} \tag{2}$$
     is a **real** fundamental set of solutions of the system $X' = AX$.

   **We are interested in real solutions, so when you are asked for the general solutions then it means the real general solutions and you have to convert** (1) **into** (2) **and take the linear combination with real coefficients of what you obtained**.

7. Example. Consider $\begin{pmatrix} 3 & 1 \\ -5 & 1 \end{pmatrix}$

   (a) Find general solution of the system $X' = AX$.

(b) Find solution subject to the initial conditions $x_1(0) = 2, x_2(0) = 3$.

8. Case $n = 3$. If $\lambda$ is complex eigenvalue of the coefficient matrix $A$, then $\bar{\lambda}$ is an eigenvalue as well and the third eigenvalue must be real (denote it by $\mu$). Let $v$ and $w$ be eigenvectors corresponding to $\lambda$ and $\mu$, respectively. Then

$$\left\{\mathrm{Re}(e^{\lambda t}v), \mathrm{Im}(e^{\lambda t}v), e^{\mu t}w\right\}$$

is a **real** fundamental set of solutions of the system $X' = AX$.

## The relation between the characteristic equation of a scalar linear differential equation with constant coefficients and the the characteristic equation of the corresponding system of fits order differential equations

9. Given a differential equation

$$ay'' + by' + cy = 0 \tag{3}$$

find the characteristic equation of the corresponding system of first order differential equations and prove that the eigenvalues of the matrix of this system coincide with the roots of the characteristic polynomial of (3).