

23: Systems of homogeneous linear equations: the case of complex eigenvalues (section 7.6)

1. Let A be an $n \times n$ matrix with real entries. It may happen that the characteristic equation $\det(A - \lambda I) = 0$ has a complex root λ , i.e. λ is a complex eigenvalue of A . Then a corresponding eigenvector v has complex components, because it satisfies the linear algebraic system of equations $(A - \lambda I)v = 0$ and the matrix $A - \lambda I$ already has complex entries on its diagonal.
2. If λ is a complex eigenvalue of the real matrix A with an eigenvector v , then $\bar{\lambda}$ (the complex conjugate of λ) is an eigenvalue of A with an eigenvector \bar{v} . In other words complex eigenvalues of a real matrix A come in pairs of complex conjugate ones.
3. Hence if λ is a complex eigenvalue of a real matrix A with an eigenvector v then both $e^{\lambda t}v$ and $e^{\bar{\lambda}t}\bar{v}$ are solutions of the system $X' = AX$. These solutions are vector-valued functions with complex components. Then, by the superposition principle, the vector-valued functions $\operatorname{Re}(e^{\lambda t}v)$ and $\operatorname{Im}(e^{\lambda t}v)$, i.e. the vector-valued functions of the real and imaginary parts of the components of $e^{\lambda t}v$ are solutions of the same system $X' = AX$.
4. Note that any vector v with complex components can be uniquely represented as $v = a + ib$ where a and b are vectors with real components, the real and imaginary parts of v respectively.
5. If $\lambda = \alpha + i\beta$, $\beta \neq 0$ is an eigenvalue of a real matrix A and $v = \mathbf{a} + i\mathbf{b}$ is an eigenvector of λ , where a and b are vectors with real components then

$e^{t\lambda}v =$ i.e.

$$\begin{aligned} \operatorname{Re}(e^{\lambda t}v) &= e^{\alpha t}(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}), \\ \operatorname{Im}(e^{\lambda t}v) &= e^{\alpha t}(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}) \end{aligned}$$

6. Case $n = 2$. If $\lambda = \alpha$ is a complex eigenvalue of the coefficient matrix A in the homogeneous system $X' = AX$ and v is a corresponding eigenvector then

- $$\{e^{\lambda t}v, e^{\bar{\lambda}t}\bar{v}\} \tag{1}$$

is a fundamental set of (complex) solutions of the system $X' = AX$.

- $$\{\operatorname{Re}(e^{\lambda t}v), \operatorname{Im}(e^{\lambda t}v)\} \tag{2}$$

is a **real** fundamental set of solutions of the system $X' = AX$.

We are interested in real solutions, so when you are asked for the general solutions then it means real general solutions and you have to convert (1) into (2) and take the linear combination of what you obtained.

7. Example. Consider $\begin{pmatrix} 3 & 1 \\ -5 & 1 \end{pmatrix}$

(a) Find general solution of the system $X' = AX$.

(b) Find solution subject to the initial conditions $x_1(0) = 2, x_2(0) = 3$.

8. Case $n = 3$. If $\alpha \pm i\beta$ are complex eigenvalue of the coefficient matrix A , then the third eigenvalue must be real (denote it by λ). Let v and w be eigenvectors corresponding to $\alpha + i\beta$ and λ , respectively. Then

$$\{\operatorname{Re}(e^{\lambda t}v), \operatorname{Im}(e^{\lambda t}v), e^{\lambda t}w\}$$

is a **real** fundamental set of solutions of the system $X' = AX$.

The relation between the characteristic equation of a scalar linear differential equation with constant coefficients and the the characteristic equation of the corresponding system of fits order differential equations

9. Given a differential equation

$$ay'' + by' + cy = 0 \tag{3}$$

find the characteristic equation of the corresponding system of first order differential equations and prove that the eigenvalues of the matrix of this system coincide with the roots of the characteristic polynomial of (3).