24: Toward the case of repeated eigenvalues: matrix exponential (section 7.7)

1. Recall that the Taylor expansion of the exponential function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{1}$$

which implies that

$$e^{rt} = 1 + tr + \frac{t^2}{2!}r^2 + \frac{t^3}{3!}r^3 + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!}r^i. \tag{2}$$

In particular, in the previous notes, section 14, item 11 there, we used this formula to prove that

$$\frac{d}{dt}e^{rt} = re^{rt} \tag{3}$$

for any complex r.

2. Now replace the number x by an $n \times n$ matrix A in (1) (replacing also the number 1 in the very first term by the $n \times n$ identity matrix I) to obtain another $n \times n$ matrix e^A called the *matrix exponential of* A:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i!}A^i \tag{4}$$

Then we can consider e^{tA} (a matrix-values function assigning to any t the matrix exponential of the matrix tA),

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!}A^i. \tag{5}$$

and exactly as for (6) one can obtain from this that

$$\frac{d}{dt}e^{tA} = Ae^{tA} \tag{6}$$

This implies that *for any column vector v the vector function $e^{tA}v$ is a solution of the system $X' = AX$.*

3. One says that n vectors $v^1, \dots, v^n$ constitute a basis in $\mathbb{R}^n$ if any other vector v in $\mathbb{R}^n$ can be uniquely represented as a linear combination of $v^1, \dots, v^n$, i.e. there exist constants $c_1, \dots, c_n$ such that

$$v = c_1 v^1 + \dots c_n v^n.$$

Equivalently, $v^1, \dots, v^n$ constitute a basis in $\mathbb{R}^n$ if and only if $\det(v^1, \dots, v^n) \neq 0$.

Based on the previous item, if $v^1, \dots v^n$ constitute a basis of $\mathbb{R}^n$, then

$$\{e^{tA}v^1, e^{tA}v^2, \dots, e^{tA}v^n\} \tag{7}$$

form a fundamental set of solutions of A

The big question: How to calculate e^{At} or how to find a convenient basis $v^1, \dots, v^n$ for which $e^{At}v^i$ can be effectively calculated for every $i = 1, \dots, n$?

4. Note that in contrast to numbers for matrices $e^A e^B \neq e^B e^A$. However, if the matrices A and B commute, $AB = BA$ then $e^A e^B = e^B e^A$ (just take the products of their Taylor expansion and use commutativity). Moreover, if $AB = BA$, then $e^{A+B} = e^A e^B$, which is not true in general.

5. If $A = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ on the diagonal, then $e^{tA} = \text{diag}\{e^{t\lambda_1}, \dots, e^{t\lambda_n}\}$

In particular, if $A = \lambda I$, A is the diagonal matrix with all entries on the diagonal being equal, then $e^{t\lambda I} = e^{\lambda t} I$

6. Note that the matrix λI commutes with any other matrix. Therefore using the previous two items we can get the following formula that will be crucial in the sequal:

$$\boxed{e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}} \tag{8}$$

7. Why the formula (8) is useful. Assume that λ is an eigenvalue and v is the corresponding eigenvector. Then calculate $e^{tA}v$ using (8)

So, $e^{tA}v = e^{\lambda t}v$ as expected by our previous considerations (see section 21, item 3.)

8. Conclusion: As a consequence of item 3 above (see the sentence including formula (7) , if an $n \times n$ matrix A admits a basis of eigenvectors $v^1, \dots v^n$ in $\mathbb{R}^n$, then $(e^{t\lambda_1}v^1, \dots, e^{t\lambda_n}v^n)$ form a fundamental set of solutions of $X' = AX$.

   Note that we could derive it without using matrix exponential and (7), but the matrix exponential gives a way that works in more general situation.

9. **The life is not so simple: not any $n \times n$ matrix admits a basis of eigenvectors in $\mathbb{R}^n$.**

   EXAMPLE 1. *Let* $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

   (a) Find all eigenvectors of N. Can you choose eigenvectors of N that constitute a basis of $\mathbb{R}^2$

   (b) Calculate e^{tN}.