

25: Repeated Eigenvalues: algebraic and geometric multiplicities of eigenvalues, generalized eigenvectors, and solution for systems of differential equation with repeated eigenvalues in case $n = 2$ (sec. 7.8)

1. We have seen that not every matrix admits a basis of eigenvectors. First, discuss a way how to determine if there is such basis or not.

Recall the following two equivalent characterization of an eigenvalue:

- (a) λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$;
- (b) λ is an eigenvalue of $A \Leftrightarrow$ there exist a nonzero vector v such that $(A - \lambda I)v = 0$.
The set of all such vectors together with the 0 vector form a vector space called the *eigenspace of λ* and denoted by E_λ .

Based on these two characterizations of an eigenvalue λ of a matrix A one can assign to λ the following two positive integer numbers,

- **Algebraic multiplicity** of λ is the multiplicity of λ in the characteristic polynomial $\det(A - xI)$, i.e. the maximal number of appearances of the factor $(x - \lambda)$ in the factorization of the polynomial $\det(A - xI)$.
 - **Geometric multiplicity** of λ is the dimension $\dim E_\lambda$ of the eigenspace of λ , i.e. the maximal number of linearly independent eigenvectors of λ .
2. For example, if $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (as in the example in item 9 of the previous notes), then $\lambda = 0$ is the unique eigenvalue. Find the algebraic multiplicity and geometric multiplicity of $\lambda = 0$

THEOREM 1. *Geometric multiplicity is not greater than algebraic multiplicity.*

THEOREM 2. *A matrix A admits a basis of eigenvectors if and only if for every its eigenvalue λ the geometric multiplicity of λ is equal to the algebraic multiplicity of λ .*

REMARK 3. *In Linear Algebra matrices admitting a basis of eigenvectors are called diagonalizable (because they are diagonal in this basis).*

REMARK 4. *Basis of eigenvectors always exists for the following classes of matrix:*

- **symmetric matrices:** $A^T = A$, or equivalently, $a_{ij} = a_{ji}$ for all i, j ;
- **skew-symmetric** $A^T = -A$, or equivalently, $a_{ij} = -a_{ji}$ for all i, j .

For symmetric matrices all eigenvalues are real and the the eigenspaces corresponding to the different eigenvalues are orthogonal. For skew-symmetric matrices the eigenvalues are purely imaginary (i.e. of the form $i\beta$).

3. If the matrix A does not admit a basis of eigenvectors then for what vectors w other than the eigenvectors it is still easy to calculate $e^{At}w$ in the light of the formula

$$\boxed{e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}} \quad (1)$$

(see item 6 of the previous lecture notes)?

4. Assume that w is such that

$$(A - \lambda I)w \neq 0, \text{ but } (A - \lambda I)^2 w = 0 \quad (2)$$

(the first relation means that w is not an eigenvector corresponding to λ). Calculate $e^{At}w$ using (1).

5. More generally if we assume that for some $k > 0$

$$(A - \lambda I)^{k-1} w \neq 0, \text{ but } (A - \lambda I)^k w = 0 \quad (3)$$

then $e^{At}w$ can be calculated using only finite number of terms when expanding $e^{t(A-\lambda I)}w$ from (1).

Note that if for some λ there exists w satisfying (3) then λ must be an eigenvalue

- 6.

DEFINITION 5. A vector w satisfying (3) for some $k > 0$ is called a generalized eigenvector of λ (of order k).

The set of all generalized eigenvectors of λ together with the 0 vector is a vector space denoted by E_λ^{gen}

REMARK 6. The (regular) eigenvector is a generalized eigenvector of order 1, so $E_\lambda \subset E_\lambda^{\text{gen}}$ (given two sets A and B , the notation $A \subset B$ means that the set A is a subset of the set B , i.e. any element of the set A belongs also to B)

THEOREM 7. The dimension of the space E_λ^{gen} of generalized eigenvectors of λ is equal to the algebraic multiplicity of λ .

THEOREM 8. Any matrix A admits a basis of generalized eigenvectors.

Let us see how it all works in the first nontrivial case of $n = 2$.

7. Let A be 2×2 matrix and λ is a repeated eigenvalue of A . Then its algebraic multiplicity is equal to _

There are two options for the geometric multiplicity:

1 (trivial case) Geometric multiplicity of λ is equal to 2. Then $A = \lambda I$

2. (less trivial case) Geometric multiplicity λ is equal to 1. In the rest of these notes we concentrate on this case only.

8.

PROPOSITION 9. *Let w be a nonzero vector which is not an eigenvector of λ , $w \notin E_\lambda$. The vector w satisfies $(A - \lambda I)^2 w = 0$, i.e. w is a generalized eigenvector of order 2? Besides, in this case*

$$v := (A - \lambda I)w \quad (4)$$

is an eigenvector of A corresponding to λ .

9. Note that $\{v, w\}$ constructed above constitute a basis of \mathbb{R}^2 (i.e. $E_\lambda^{\text{gen}} = \mathbb{R}^2$, so we proved Theorem 8 in this case. Therefore, $\{e^{tA}v, e^{tA}w\}$ form a fundamental set of solutions for the system $X' = AX$. By constructions and calculation as in item 4 above

$$e^{tA}v =$$

$$e^{tA}w =$$

Conclusion:

$$\boxed{\{e^{\lambda t}v, e^{\lambda t}(w + tv)\}}. \quad (5)$$

form a fundamental set of solutions of $X' = AX$, i.e. the general solution is

$$\boxed{e^{\lambda t}(C_1v + C_2(w + tv))}. \quad (6)$$

10. This gives us the following algorithms for finding the fundamental set of solutions in the case of a repeated eigenvalue λ with geometric multiplicity 1.

Algorithm 1 (easier than the one in the book):

- (a) Find the eigenspace E_λ of λ by finding all solutions of the system $(A - \lambda I)v = 0$. The dimension of this eigenspace under our assumptions must be equal to $_$.
- (b) Take any vector w not lying in the eigenline E_λ and find $v := (A - \lambda I)w$. With chosen v and w the general solution is given by (6).

Algorithm 2 (as in the book):

- (a) Find an eigenvector v by finding one nonzero solution of the system $(A - \lambda I)v = 0$.
- (b) With v found in item 1 find w such that $(A - \lambda I)w = v$. With chosen v and w the general solution is given by (6).

REMARK 10. *The advantage of Algorithm 1 over Algorithm 2 is that in the first one you solve only one linear system when finding the eigenline, while in Algorithm 2 you need to solve one more linear system $(A - \lambda I)w = v$ for w (in Algorithm 1 you choose w and then find v from (4) instead).*

11. Finally let us give another algorithm which works only in the case $n = 2$ (for higher n it works only under some additional assumption that A has only one eigenvalue). This algorithm does not use eigenvectors explicitly (although implicitly we use here the information that an eigenvalue λ is repeated). Proposition 9 actually implies that

$$(A - \lambda I)^2 = 0. \quad (7)$$

then based on (1) and (7)

$$e^{tA} =$$

Conclusion:

$$e^{At} = e^{\lambda t}(I + t(A - \lambda I)) \quad (8)$$

Algorithm 3: Calculate e^{tA} from (8). The columns of the resulting matrix form a fundamental set of solutions.

REMARK 11. *Identity (7) is in fact a particular case of the following remarkable result from Linear Algebra, called Caley-Hamilton: Let*

$$\det(A - \lambda I) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0) \quad (9)$$

then

$$A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I = 0 \quad (10)$$

In other words if one substitutes the matrix A instead of λ and a_0I instead of a_0 into the right hand side of (10) then you will get 0.

12. Example. Find general solution of the system.:
$$\begin{cases} x_1' &= -3x_1 + \frac{5}{2}x_2 \\ x_2' &= -\frac{5}{2}x_1 + 2x_2 \end{cases}$$