26: Repeated Eigenvalues: description of general case with applications to $n = 3$ with an eigenvalue of algebraic multiplicity 2 (sec. 7.8)

1. In the previous notes we defined the notions of the algebraic and geometric multiplicity of an eigenvalue (see item 1) and of generalized eigenvector of order k (Definition 5). We also saw (item 5) that it is relatively easy to calculate $e^{tA}w$ for a generalized eigenvector w of any order using the formula

$$\boxed{e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}} \tag{1}$$

and we stated that any matrix admits a basis of generalized eigenvectors (Theorem 8).

Now I will give more properties of (spaces of) generalized eigenvectors so that you will have more clear idea how to find a basis of them in an effective way and I will demonstrate this in all possible cases when $n = 3$.

2. Let λ be an eigenvalue of matrix A. For any positive integer k let $E_\lambda^{(k)}$ be the space of all vectors w in $\mathbb{R}^n$ such that

$$(A - \lambda I)^k w = 0.$$

Some comments:

- In set-theoretical notations $E_\lambda^{(k)}$ can be defined without words, just by the formula:

$$E_\lambda^{(k)} := \{w : (A - \lambda I)^k w = 0\};$$

- By Definition 5 , $E_\lambda^{(k)}$ is the space consisting of all generalized eigenvectors of order not greater than k (and the zero vector);
- the space $E_\lambda^{(1)}$ is nothing but the eigenspace E_λ of λ.

3. Some main properties of $E_\lambda^{(k)}$

   (a) $E_\lambda^{(k)} \subset E_\lambda^{(k+1)}$, so we have the following nested sequence of subspace (called also a filtration of $\mathbb{R}^n$):

   (b) If $E_\lambda^{(k)} = E_\lambda^{(k+1)}$ then $E_\lambda^{(k)} = E_\lambda^{(n)}$ for all $n > k$, i.e. the nested sequence of subspaces stabilizes on the kth step and in this case $\dim E_\lambda^{(k)} =$ the algebraic multiplicity of λ (so in this case $E_\lambda^{(k)}$ is the space E_λ^{gen} consisting of all generalized eigenvectors (and zero vector) in the notation of the previous notes.

(c) $(A - \lambda I)E_\lambda^{(k)} \subset E_\lambda^{(k-1)}$. Moreover, if w is a generalized eigenvector of order k then $(A - \lambda I)w$ is a generalized eigenvector of order $k - 1$.

Let us demonstrate how to use these properties in all possible situations that may occur in the case of repeated eigenvalues when $n = 3$. In this notes we demonstrate it in the case when there is an eigenvalue λ_1 of algebraic multiplicity 2 and in the next set of notes we will discuss the case of an eigenvalue of algebraic multiplicity 3.

$n = 3$: there is an eigenvalue λ_1 of algebraic multiplicity 2

4. Then another eigenvalue λ_2 has algebraic multiplicity equal to __ and so its geometric multiplicity is equal to __.

5. λ_1, having algebraic multiplicity 2, may have either geometric multiplicity ___ or ___

6. If λ_1 has geometric multiplicity 2 then by Theorem 2 of the previous notes the matrix A has a basis of eigenvectors. So, you proceed as follows:
   - Find 2 linearly independent eigenvectors v^1 and v^2 of the eigenvlue λ_1 (we will demonstrate the technique how to do it in Example 1 below, see also Remark 1 below);
   - Find an eigenvector z of λ_2;
   - The general solution of $X' = AX$ is

     $$X(t) = C_1 e^{\lambda_1 t} v^1 + C_2 e^{\lambda_1 t} v^2 + C_3 e^{\lambda_2} z$$

     similarly to the case of distinct eigenvalues.

7. If λ_1 has geometric multiplicity 1, then
   - $\dim E_{\lambda_1}^{(1)} = \dim E_{\lambda_1} =$ ___.
   - Then by stabilization property (b) of item 3 above the space $E_{\lambda_1}^{(2)}$ is strictly larger than the eigenspace E_{λ_1}, so $\dim E_{\lambda_1}^{(2)} > 1$. Also since the algebraic multiplicity of λ_1 is 2, by Theorem 7 of the previous notes $\dim E_{\lambda_1}^{(2)} \leq 2$. So, taking into account both inequalities for $\dim E_{\lambda_1}^{(2)}$ we get that $\dim E_\lambda^{(2)} = 2$.

Based on this we can proceed in one of the following ways

8. **Analog of the algorithm 1 of the previous notes**

   (a) Find the eigenspace E_{λ_1} by solving the system $(A - \lambda_1 I)v = 0$. If it is one-dimensional then we are in the situation discussed here (if it is two dimensional then go to the item 6 above);

   (b) Find the space $E_{\lambda_1}^{(2)}$ by solving the system $(A - \lambda_1 I)^2 w = 0$ (it might be quite tedious to calculate the matrix $(A - \lambda_1 I)^2$ but the good news that you know a priori that all rows of this matrix must be multiple of one row);

   (c) Choose any vector w in the plane $E_{\lambda_1}^{(2)}$, which does not lie on the eigenline E_{λ_1} and set $v := (A - \lambda_1 I)w$. Then, by property (c) of item 3 above, the vector v is an eigenvector with the eigenvalue λ_1;

   (d) Find an eigenvector z of the second eigenvalue λ_2;

   (e) Then $\{v, w, z\}$ is a basis of $\mathbb{R}^3$ consisting of 2 eigenvectors (v and z) and one generalized eigenvector w. So, similarly to item 9 of the previous notes:

   $$(e^{tA}v, e^{tA}w, e^{tA}z) = \{e^{\lambda_1 t}v, e^{\lambda_1 t}(w + tv), e^{\lambda_2 t}z\} \tag{2}$$

   is the fundamental set of solutions of $X' = AX$.

9. **Analog of the algorithm 2 of the previous notes (as in the textbook)**

   (a) Solve the system $(A - \lambda_1 I)v = 0$. If the space of solutions is one-dimensional, then we are in the situation discussed here (if it is two dimensional then go to the item 6 above). Choose one of the solutions v of the system $A - \lambda_1 I)v = 0$.

   (b) Find a vector w such that $(A - \lambda_1 I)w = v$, where v is the vector chosen in the previous step (the advantage here over the previous algorithm is that you do not need to calculate the matrix $(A - \lambda_1 I)^2$).

   (c) With the vectors v and w found in the previous items proceed as in the items (d) and (e) of the previous algorithm.

   REMARK 1. **Useful technical remark** *the very first thing here is to determine the geometric multiplicity of λ_1. Depending on this you will proceed either as in item 6 or either items 8-9. Some people like the following rule for determining geometric multiplicity:* bring the matrix $A - \lambda_1 I$ to the row echelon form using the Gauss elimination, then the geometric multiplicity is equal to number of zero rows in this row echelon form.

10. Example 1. *Consider the following system*

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 + 4x_3 \\ x_2' &= 2x_1 + 2x_3 \\ x_3' &= 4x_1 + 2x_2 + 3x_3 \end{aligned}$$

It is know that the characteristic polynomial of the matrix A of the system is equal to $-(\lambda+1)^2(\lambda-8)$.

(a) Find algebraic and geometric multiplicities of all eigenvalues of A

(b) Find the general solution of the system.

11. Example 2. *Consider the following system*

$$\begin{aligned} x_1' &= -3x_1 + 3x_2 + 8x_3 \\ x_2' &= 11x_1 - 4x_2 - 17x_3 \\ x_3' &= -5x_1 + 3x_2 + 10x_3 \end{aligned}$$

It is know that the characteristic polynomial of the matrix A of the system is equal to $-(\lambda - 2)^2(\lambda + 1)$.

(a) Find algebraic and geometric multiplicities of all eigenvalues of A

(b) Find the general solution of the system.