27: Repeated Eigenvalues continued: $n = 3$ with an eigenvalue of algebraic multiplicity 3 (discussed also in problems 18-19, page 437-439 of the book)

1. We assume that 3×3 matrix A has one eigenvalue λ_1 of algebraic multiplicity 3. It means that there is no other eigenvalues and the characteristic polynomial of a is equal to ________.

   Also we have the following three options for geometric multiplicities of λ_1: ___, ___, or ___.

   REMARK 1. *If among all 3×3 matrices with an eigenvalue with multiplicity 3 one picks up a matrix randomly, then with probability 1 the geometric multiplicity of this eigenvalue will be equal to 1.*

   REMARK 2. *For a system of first order linear homogeneous equation that comes from a scalar linear homogeneous equation of higher order, the geometric multiplicity of any eigenvalue is equal to 1. This also shows that the theory of linear systems of first order is more general than the theory of scalar eauations of higher order: not every system comes from a single higher order equation.*

2. Actually the considered case can be immediately treated by the Cayley-Hamilton Theorem by analogy with Algorithm 3 for $n = 2$: the characteristic polynomial of A is equal to $-(x - \lambda_1)^3$ so

   $$(A - \lambda I)^3 = 0$$

   Using this we have that

   $$e^{tA} = e^{\lambda_1 t} e^{t(A-\lambda_1 I)} = e^{\lambda_1 t}\big(I + t(A - \lambda_1 I) + \frac{t^2}{2}(A - \lambda_1 I)^2\big) \tag{1}$$

   Then the columns of the resulting matrix form a fundamental set of solutions.

   REMARK 3. *As a matter of fact if the geometric multiplicity of λ_1 is greater than 1, then the formula (1) can be simplified:*

   - *if geometric multilicity of λ_1 is 3, then $A = \lambda_1 I$, i.e. $A - \lambda_1 I = 0$ so (1) simplifies to*

     $$e^{tA} = e^{\lambda_1 t} I; \tag{2}$$

   - *if geometric multilicity of λ_1 is 2, then $A - \lambda_1 I \neq 0$ but $(A - \lambda_1 I)^2 = 0$, so (1) simplifies to*

     $$e^{tA} = e^{\lambda_1 t}\big(I + t(A - \lambda I)\big); \tag{3}$$

     $(A - \lambda_1 I) w_1 = v$

- *if geometric multilicity of λ_1 is* 1*, then* $(A-\lambda_1 I)^2 \neq 0$ *(but* $(A-\lambda_1 I)^3 = 0$*), so* (1) *cannot be simplified.*

However, the method of the item above will not work if one has an eigenvalue of algebraic multilicity 3 for $n > 3$, therefore I would like to present the scheme that uses eigenspaces and generalized eigenspaces (i.e. analogous to Algorithms 1 and 2).

3. The case when λ_1 has the geometric multiplicity equal to 3 is simple. In this case the eigenspace $E_{\lambda_1} = \mathbb{R}^3$, so $A = \lambda_1 I$ which means that $A = \lambda_1 I$ and so the formula (2) holds.

4. The case when λ_1 has the geometric multiplicity equal to 2:

   - In this case $\dim E_{\lambda_1} =$ ___
   - Then $\dim E^{(2)}_{\lambda_1} =$ __
     Explanation:

   - Therefore we have the following analog of Algorithm 1:

     (a) Suppose we found E_{λ_1} and it is 2-dimensional. Then take any w which is not in E_{λ_1}. Then w is a generalized eigenvector of order __. Set
     $v_1 = (A - \lambda_1 I)w$
     Then v_1 is a generalized eigenvector of order __, i.e. v_1 is an eigenvector of λ_1.

     (b) Choose another eigenvector v_2 of λ_1 which is not collinear to v_1, so that v_1 and v_2 form a basis of the eigenspace E_{λ_1}.

     (c) Then $\{v_1, w, v_2\}$ form a basis of generalized eigenvectors in $\mathbb{R}^3$ (with v_1 and v_2 being eigenvectors and w being a generalized eigenvector of order 2) and so

     $$(e^{tA}v_1, e^{tA}w, e^{tA}v_2\} = \{e^{\lambda_1 t}v_1, e^{\lambda_1 t}(w + tv_1), e^{\lambda_1 t}v_2\}$$

     form a fundamental set of solutions of $X' = AX$.

- You can try to build an analog of Algorithm 2 here (see also discussions in Problem 19 on page 438-439 of the book): First find the eigenspace E_{λ_1} , which is a plane. The problem here that, if we choose some eigenvector v, the system

$$(A - \lambda_1 I)w = v \tag{4}$$

does not have necessarily a solution, so you need to choose a specific v in E_{λ_1} for which (4) has a solution (see item (c) of Problem 19, page 439). **In contrast, Algorithm 1 does not have this problem at all, so it is certainly more efficient in this and more general situations than the Algorithm 2**.

5. The case when λ_1 has the geometric multiplicity equal to 1:

- In this case $\dim E_{\lambda_1} = \_\_\_$
- Then $\dim E^{(2)}_{\lambda_1} = \_\_$

- Then $\dim E^{(3)}_{\lambda_1} = \_\_$, so $E^{(3)}_{\lambda_1} = \mathbb{R}^3$
- Analog of Algorithm 1:
  (a) Suppose we already found the spaces E_{λ_1} (by solving the system $(A - \lambda_1 I)v = 0$) and it is 1-dimensional as expected. Then we find the space $E^2_{\lambda_1}$ by solving the system $(A - \lambda_1 I)^2 w = 0$. $E^2_{\lambda_1}$ must be 2-dimensional;
  (b) Choose any vector w_2 which is not in the plane $E^2_{\lambda_1}$. Then w_2 is the generalized eigenvector of order __. Then set

$$w_1 := (A - \lambda_1 I)w_2, \quad v = (A - \lambda_1 I)w_1$$

  (c) Then w_1 is a generalized eigenvector of order __ and v is a generalized eigenvector of order __. Also, vectors $\{v, w_1, w_2\}$ form a basis of $\mathbb{R}^3$ and
  $e^{tA}v =$
  $e^{tA}w_1 =$
  $e^{tA}w_2 =$
  So

$$\{e^{\lambda_1}v, e^{\lambda_1 t}(w_1 + tv), e^{\lambda_1 t}(w_2 + tw_1 + \frac{t^2}{2}v\} \tag{5}$$

  form a fundamental set of solutions of the system $X' = AX$.

- Analog of Algorithm 1:
  (a) Choose an eigenvector v (all eigenvector are collinear one to each other);
  (b) Find w_1 such that $(A - \lambda_1 I)w_1 = v$;
  (c) Find w_2 such that $(A - \lambda_1 I)w_2 = w_1$;
  (d) Proceed as in the item (c) of the previous algoritm.

  REMARK 4. *In general the geometric and algebraic multiplicity are not the only characteristics of an eigenvalue as in the small dimensional examples we discusssed. In higher dimensions there are additional characteristics related to partitions of an algebraic multiplicities into the sum of positive integers, then the geometric multiplicity is the number of terms in the partition. This is related to the theory of elementary divisors.*