Notes of class of 04/20/2017 MATH 308

Phase plane continued (section 9.1)

$$X' = AX, \quad X \in \mathbb{R}^2 \qquad (1)$$

Case 2 Complex eigenvalues

Assume that eigenvalues are $\alpha \pm i\beta$, $\beta \neq 0$

For definiteness take the eigenvalue $\lambda = \alpha + i\beta$ with $\beta > 0$ and let $v = a + ib$ be a corresponding eigenvector

Recall that to find a fundamental set of solutions we calculate the real and imaginary part of $e^{\lambda t} v$:

$$e^{\lambda t} v = e^{\alpha t}(\cos\beta t + i \sin\beta t)(a + ib) = e^{\alpha t}(\cos\beta t\, a - \sin\beta t\, b) +$$
$$+ i\, e^{\alpha t}(\sin\beta t\, a + \cos\beta t\, b) \Rightarrow \text{the general solution is}$$

$$X(t) = C_1 e^{\alpha t}(\cos\beta t\, a - \sin\beta t\, b) + C_2 e^{\alpha t}(\sin\beta t\, a + \cos\beta t\, b) =$$
$$= \underbrace{e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)}_{y_1(t)} a + \underbrace{e^{\alpha t}(-C_1 \sin\beta t + C_2 \cos\beta t)}_{y_2(t)} b$$

So, in coordinates (y_1, y_2) with respect to the basis (a, b) the integral curves of system (1) are given by the following parametric equation:

$$\begin{cases} y_1(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) \\ y_2(t) = e^{\alpha t}(-C_1 \sin\beta t + C_2 \cos\beta t) \end{cases}$$

Page 2) Let as before (section 3.7)

$$R = \sqrt{c_1^2 + c_2^2} \text{ and } \delta \text{ is such that}$$

$$\cos\delta = \frac{c_1}{R}, \quad \sin\delta = \frac{c_2}{R}$$

$$\Rightarrow \begin{cases} y_1(t) = e^{\alpha t} R(\cos\delta\cos\beta t + \sin\delta\sin\beta t) = e^{\alpha t} R\cos(\beta t - \delta) \\ y_2(t) = e^{\alpha t} R(-\cos\delta\sin\beta t + \sin\delta\cos\beta t) = -e^{\alpha t} R\sin(\beta t - \delta) \end{cases}$$

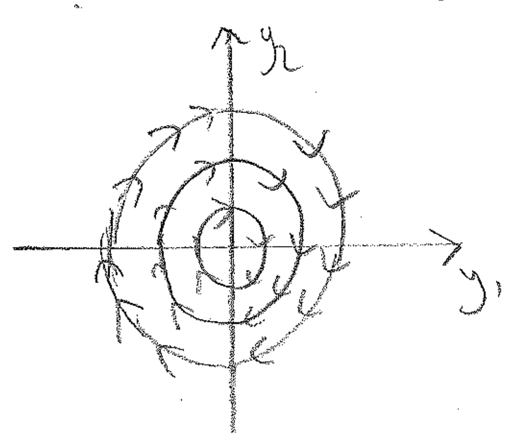
or shortly

$$\begin{cases} y_1(t) = e^{\alpha t} R\cos(\beta t - \delta) \\ y_2(t) = -e^{\alpha t} R\sin(\beta t - \delta) \end{cases} \qquad (2)$$

$\boxed{\text{Case a)}}$ $\alpha = 0$ ($\Leftrightarrow \operatorname{Re}\lambda = 0$)

$$(2) \Leftrightarrow \begin{cases} y_1(t) = R\cos(\beta t - \delta) \\ y_2(t) = -R\sin(\beta t - \delta) \end{cases} \Rightarrow y_1^2 + y_2^2 = R^2$$

This is the parametric equation of the circle of radius R in y_1y_2-plane with direction of motion clockwise (with angular velocity β)



Returning to the original coordinates (x_1, x_2) circles are transformed to ellipses:

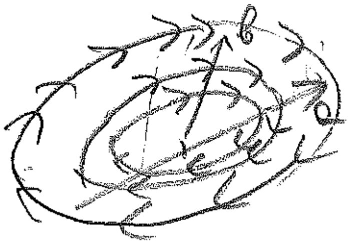


Fig 1

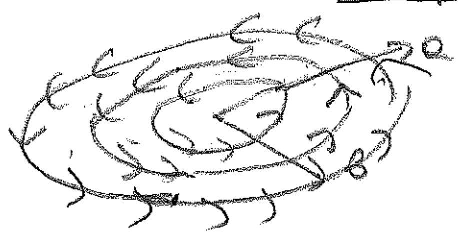


Fig 2

The origin in this case is called center and it is stable but not asymptotically stable

Direction of motion : from b to a in the shortest way (because b corresponds to the positive direction of y_2-axis and a corresponds to the positive direction of y_1-axis in y_1y_2-plane)

$\boxed{\text{Case b)}}$ $\boxed{\alpha < 0}$ $\Leftrightarrow \operatorname{Re}\lambda > 0$ → the amplitude is decaying exponentially

⇒ circles become spirals with direction of motion from b to a in the shortest way and toward the origin.

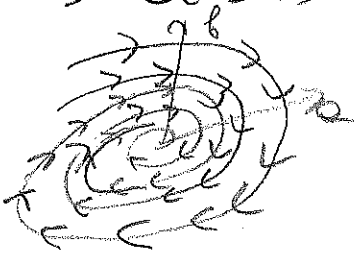


Fig 1'

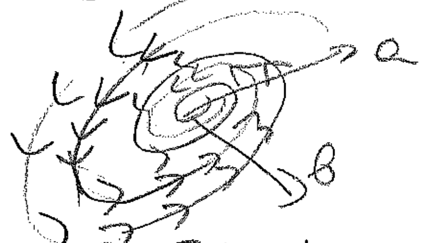


Fig 2'

→ In this case the origin is called spiral sink or stable focus. The origin is asymptotically stable

Case c) $\alpha > 0 \iff \mathrm{Re}\,\lambda > 0 \to$ the amplitude increases exponentially $\Rightarrow$ circles become spirals with direction of motion from b to a in shortest way and out of the origin

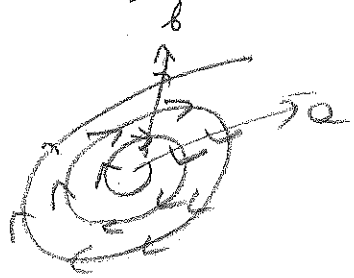


Fig 1''

Fig 2''

In this case the origin is called a spiral source or unstable focus and it is unstable

Remark 1 In contrast to eigenlines in the case of distinct real eigenvalues the vectors a and b do not have a geometric meaning: one can multiply the vector $v = a + ib$ by any complex number $z = re^{i\varphi}$ to get another eigenvector $\tilde{v} = \tilde{a} + i\tilde{b}$.

$$\tilde{a} + i\tilde{b} = z(a+ib) = r(\cos\varphi + i\sin\varphi)(a+ib)$$

$$= r\left((\cos\varphi\, a - \sin\varphi\, b) + i(\sin\varphi\, a + \cos\varphi\, b)\right).$$

$$\tilde{a} = r(\cos\varphi\, a - \sin\varphi\, b) \qquad (3)$$
$$\tilde{b} = r(\sin\varphi\, a + \cos\varphi\, b)$$

However, the direction of shortest rotation from b to a and from $\tilde{b}$ to $\tilde{a}$ is the same, which also follows from the fact that $\begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix} = 1 > 0$.

(For those who studied Linear Algebra the pair (a, b) with $a + ib$ being an eigenvector of λ defines the unique, up to a multiplication by a positive scalar, inner product in $\mathbb{R}^2$ for which (a, b) form an orthonormal basis)

Rem 2 About more exact information about the shape of ellipses/spirals

i) if $a \perp b$ then vectors a and b generate axes of symmetries of the ellipse and $|a|$ and $|b|$ are semiaxes of this ellipse

Spirals are stretched/compressed accordingly.

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ii) If b is not perpendicular to a, consider the following 2×2 symmetric matrix

$$\Gamma = \begin{pmatrix} a\cdot a & a\cdot b \\ a\cdot b & b\cdot b \end{pmatrix}$$, called the Gram matrix of (a, b).

Then among all possible $(\tilde{a}, \tilde{b})$ with $\tilde{a} + i\tilde{b}$ being an eigenvector of our original matrix A corresponding to λ one can choose a pair $(\tilde{a}, \tilde{b})$ with $\tilde{a} \perp \tilde{b}$ and in this case $\tilde{a}$ and $\tilde{b}$ are eigenvector of the Gram matrix Γ

Then with this $(\tilde{a}, \tilde{b})$ we are in the situation of item i) above.

In practice, to find such $\tilde{a}$ & $\tilde{b}$ find eigenvectors of Γ, say $\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$. Then $\alpha_1 a + \beta_1 b$ and $\alpha_2 a + \beta_2 b$ define directions of $\tilde{a}$ and $\tilde{b}$ i.e axis of symmetry of the ellipse and the ratio of semiaxis = the square root of the ratio of the eigenvalues of Γ.

Simple rule to determine the direction of motion (clockwise or counterclockwise) without the calculation of an eigenvector $v = a + ib$

Assume that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has complex roots. Then (1) if $a_{21} < 0$ then the motion is clockwise and if $a_{21} > 0$ then the motion is counterclockwise

(2) Similarly, if $a_{12} > 0$ then the motion is clockwise and if $a_{12} < 0$ then the motion is counterclockwise.

Proof of the statement about a_{12}:

Let $\lambda = \alpha + i\beta$, $\beta > 0$ is an eigenvalue of A. Then an eigenvector v satisfies

$$\begin{pmatrix} a_{11} - \alpha - i\beta & a_{12} \\ a_{21} & a_{22} - \alpha - i\beta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

and this system of 2 equations must be equivalent just to one of this equation.

For example, it is equivalent to

$$a_{21} v_1 + (a_{22} - \alpha - i\beta) v_2 = 0$$

As v we can take for example

$$v = \begin{pmatrix} -a_{22} + \alpha + i\beta \\ a_{21} \end{pmatrix} = \underbrace{\begin{pmatrix} -a_{22} + \alpha \\ a_{21} \end{pmatrix}}_{a} + i \underbrace{\begin{pmatrix} \beta \\ 0 \end{pmatrix}}_{b}$$

The shortest direction of rotation from b to a is clockwise $\Leftrightarrow$ the shortest direction of rotation from a to b is counterclockwise $\Leftrightarrow$

$$\begin{vmatrix} -a_{22} + \alpha & \beta \\ a_{21} & 0 \end{vmatrix} > 0$$

(remember the right hand rule for determining the direction of cross product)

$\Leftrightarrow -a_{21}\beta > 0 \underset{\text{since } \beta > 0}{\Leftrightarrow} a_{21} < 0$

Similarly, the shortest direction of rotation from b to a is counterclockwise $\Leftrightarrow a_{21} > 0$

The statement about the sign of a_{12} can be proved by complete analogy (try!)

3. The case of repeated eigenvalues

Subcase 3a) Geom. multiplicity of λ = Alg. multiplicity of λ = 2 $\Rightarrow$ $A = \lambda I$ i.e.

$$\begin{cases} x_1' = \lambda x_1 \\ x_2' = \lambda x_2 \end{cases} \Rightarrow \begin{matrix} x_1(t) = c_1 e^{\lambda t} \\ x_2(t) = c_2 e^{\lambda t} \end{matrix} \Leftrightarrow$$

$$\left.\begin{matrix} \text{If } c_1 \neq 0 & x_2 = \frac{c_2}{c_1} x_1 \\ \text{If } c_1 = 0 & x_1 = 0 \end{matrix}\right\} \to \text{straight line through the origin}$$

In this case the origin is called **proper node** or **star point**

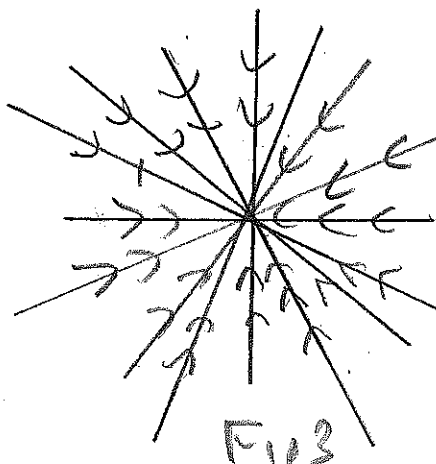


Fig 3

$\leftarrow$ If $\lambda < 0 \to$ the direction of motion is toward the origin $\to$ asymptotically stable

$\leftarrow$ If $\lambda > 0 \to$ the direction of motion is out of the origin

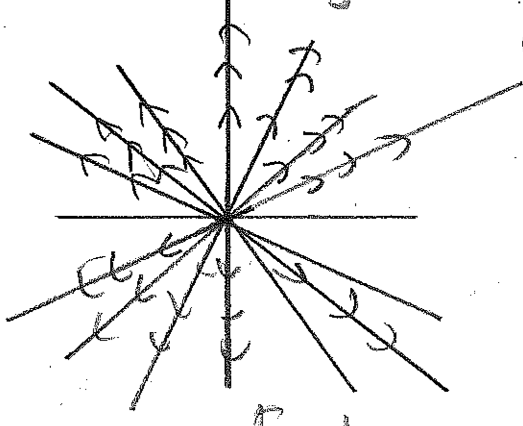


Fig 4

Subcase 3b) Geometric multiplicity of λ is 1 ($<$ algebraic multiplicity, which is 2)

Let (v, w) are such that $v = (A - \lambda I) w$, $v \neq 0$

(So, in particular, v is an eigenvector of A)

Then the general solution is

$$x(t) = e^{\lambda t}\left(C_1 v + C_2 (w + t v)\right) =$$

$$= \underbrace{e^{\lambda t}(C_1 + C_2 t)}_{y_1(t)} v + \underbrace{C_2 e^{\lambda t}}_{y_2(t)} w$$

So, in coordinates (y_1, y_2) with respect to the basis (v, w) the integral curves of the system $x' = Ax$ are given by the following parametric equation:

$$\begin{cases} y_1(t) = e^{\lambda t}(C_1 + C_2 (t)) & (4.1) \\ y_2(t) = C_2 e^{\lambda t} & (4.2) \end{cases}$$

Express y_1 as a function of y_2 by eliminating t:

Assume first that $\lambda > 0$

Also assume that $C_2 > 0 \Rightarrow y_2 > 0$

Page 11/

Express t via y_2 from equation 4.2:

$$y_2 = C_2 e^{\lambda t} \Rightarrow e^{\lambda t} = \frac{y_2}{C_2} \Rightarrow t = \frac{1}{\lambda}\ln\frac{y_2}{C_2} \Rightarrow$$

(substituting to (4.1)):

$$y_1 = \frac{y_2}{C_2}\left(C_1 + \frac{C_2}{\lambda}\ln\frac{y}{C_2}\right) = \frac{y_2}{C_2}\left(C_1 - \frac{C_2}{\lambda}\ln C_2 + \frac{C_2}{\lambda}\ln y_2\right) = \underbrace{y_2(\alpha + \beta\ln y_2)}_{f(y_2)} \quad \text{where } \beta = \frac{1}{\lambda}, \quad \alpha = \frac{C_1}{C_2} - \frac{1}{\lambda}\ln C_2$$

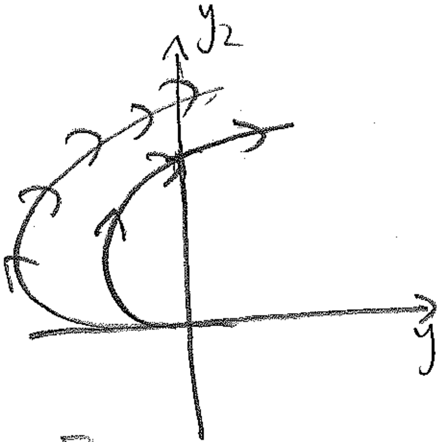


Fig. 5

One critical point for $y_2 > 0$ and since $f''(y_2) = \frac{\beta}{y_2}$ and $y_2 > 0$, $\beta > 0$ this point is a local minimum. This justifies the graphs in Fig 5. Also the direction of motion is out of the origin, because $\lambda > 0$.

In the same way if $C_2 < 0$, $y_2 < 0 \Rightarrow$ critical point of f is local maximum and the picture of the integral curve in $y_1 y_2$-plane is

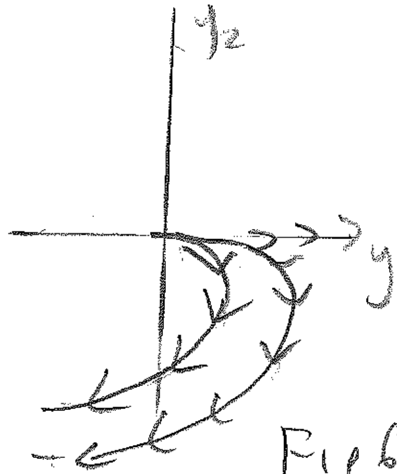
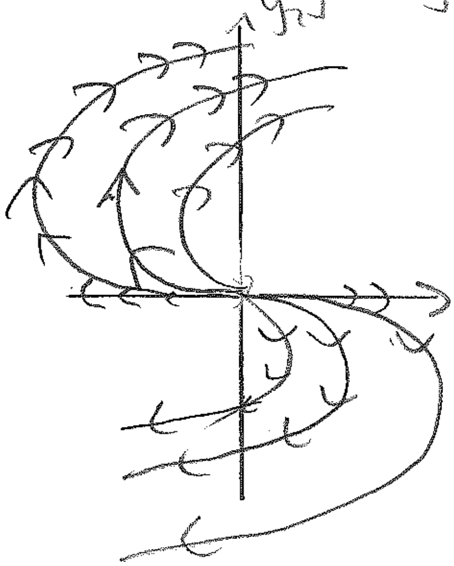


Fig 6

Combining Fig 5 & 6 we get



(note that for $c_2 = 0$
$y_2 = 0$ → the motion
of y_1-axis out of the
origin)

Returning to the original plane v corresponds to positive direction of y_1 and w corresponds to positive direction of w

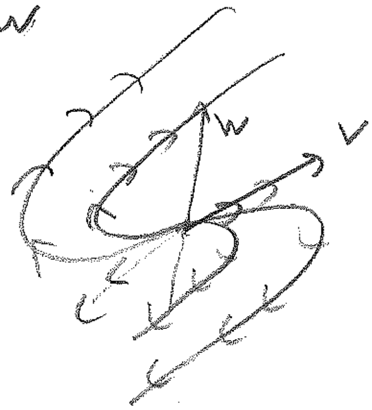
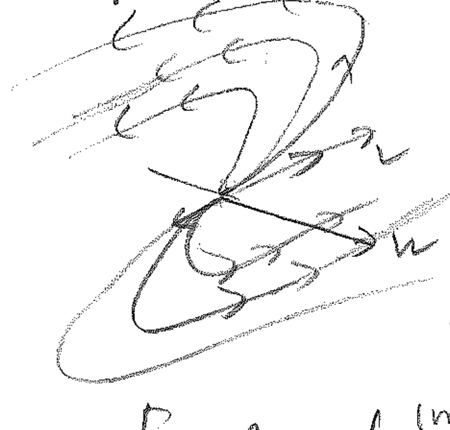


Fig 7



In this case the origin is called <u>improper node</u>

Fig 8 (improper nodal source)
→ <u>unstable</u>

Note that the vector w does not have a geometric meaning; you can replace it by $w + cv$ for any constant c, because if $v = (A - \lambda I)w$ then also $v = (A - \lambda I)(w + cv)$ (recall that $(A - \lambda I)v = 0$)

The integral curve go out of the origin such that near the origin their tangent lines are almost tangent to the Eigenline (i.e the line generated by v).

In order to understand how the trajectories go out of the origin, we can use the following rule based on our previous analysis:

The pair of vectors (v, w) divide the plane into 4 quadrants that can be enumerated as 1^{st}, 2^{nd}, 3^{rd} and 4^{th} similarly to the standard 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} quadrant:

1^{st} quadrant w.r.t. (v, w) consists of all points with coordinates (y_1, y_2) w.r.t (v, w) s.t $y_1 > 0, y_2 > 0$

2nd quadrant consist of points with $y_1<0, y_2>0$

3rd —||—||—||—|| with $y_1<0, y_2<0$

4th —||—||—||—|| with $y_1>0, y_2<0$

(see Fig 9 & 10)

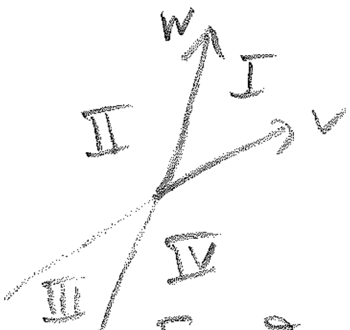


Fig. 9

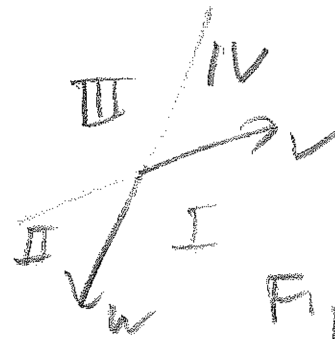


Fig. 10

Note that if we change sign of v : $v \to -v$ then also we have to change sign of w, e.g. $w \to -w$ (because if $v=(A-\lambda I)w$ then $-v=(A-\lambda I)(-w)$)

And in this case I & III quadrant swaps and II & IV quadrant swaps.

So for $\lambda>0$ the rule is: The integral curves leave the origin toward the 2nd and 4th quadrants (see fig. 7 & 8)

- Now consider the case $\lambda<0$

15) In this case we can do analysis similar to page 12 to get the following pictures

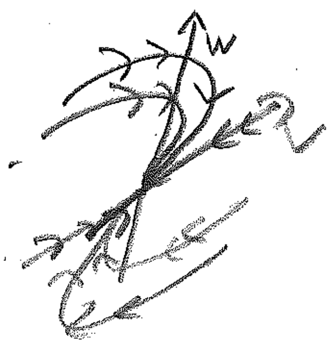


Fig 11

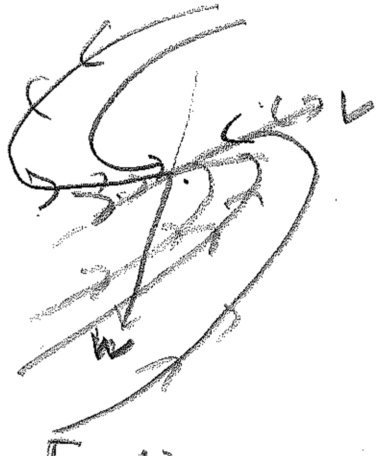


Fig 12

In this case the origin is called Improper node (Improper nodal sink) $\rightarrow$ asymptotically stable

The integral curves enter the origin with the tangent lines almost parallel to the eigenline.

The rule for how they enter: The integral curves enter the origin from the 1st and 3rd quadrant w.r.t v & w

A simple rule to determine how the phase portrait looks like without calculating w:

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has the repeated eigenvalue λ with geometric multiplicity 1

Then if $a_{21} < 0$ ($or\ a_{12} \geq 0$), then the direction of motion on a part of the integral curve which is far from the origin is clockwise as in Fig 7 (p. 12) and Fig 11

If $a_{21} > 0$ (or $a_{12} < 0$), then the direction of motion on a part of the integral curve which is far from the origin is counterclockwise as in Fig 8 (p. 12) and Fig. 12

(Note that a_{11} & a_{12} cannot be simultaneously 0, otherwise $A = I$)

<u>Proof</u>, concerning sign of a_{12}

Take $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $a_{21} \neq 0$ then w is not an eigenvector

Let $v = (A - I)w = \begin{pmatrix} a_{11} - 1 \\ a_{21} \end{pmatrix}$

Then $\det(v, w) = \begin{vmatrix} a_{11} - 1 & 1 \\ a_{21} & 0 \end{vmatrix} = -a_{21}$

and the conclusion for $a_{21} < 0$ easily follow from Fig 7 and 11 and for $a_{21} > 0$ from Fig 8 & 12.

17) The case of nonhomogeneous linear system of type $x' = Ax + \underbrace{b}_{\text{constant}}$ (5) → it appears often in control theory

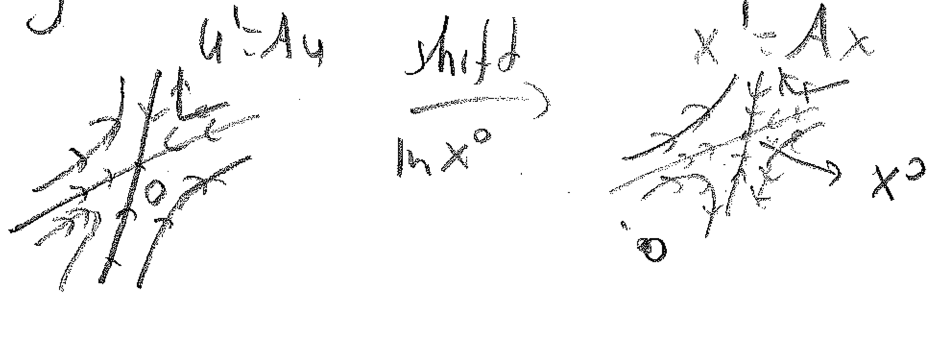
If $\det A \neq 0$ there is a unique critical point x^0, because it satisfies $Ax + b = 0 \Leftrightarrow Ax = -b$ and this system has a unique solution if $\det A \neq 0$.

$\Leftrightarrow b = -Ax_0$

(5) $\Leftrightarrow x' = Ax - Ax^0 \Leftrightarrow (x - x^0)' = A(x - x^0)$

Substitute $u = x - x^0$ to obtain the homogeneous linear system $u' = Au$ (6)

If $u(t)$ is a solution of (6) $\Leftrightarrow x(t) = x^0 + u(t)$ is a solution of (1) $\Rightarrow$ the phase portrait for (5) is obtained from the phase portrait of (6) by the shift (the translation) in x^0:



18 **Critical damping as a <u>bifurcation</u>** (qualitative change of behaviour under small change of a parameter)

Damped unforced vibration is given by

$$m u'' + \gamma u' + k u = 0 \iff \begin{cases} x_1' = x_2 \\ x_2' = -\frac{k}{m} x_1 - \frac{\gamma}{m} x_2 \end{cases}$$

$\lambda_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$

for $\gamma = 2\sqrt{mk}$ → the origin is improper node

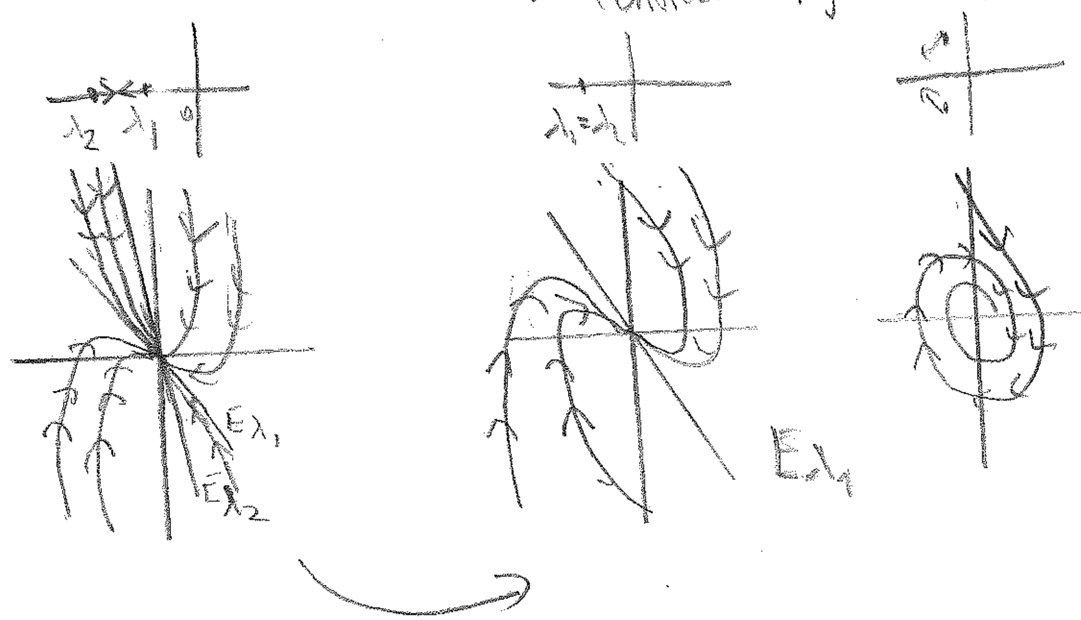
$\gamma < 2\sqrt{mk}$ → the origin is spiral sink

$\gamma > 2\sqrt{mk}$ → the origin is nodal sink

Values of γ | $\gamma > 2\sqrt{mk}$ (overdamping) | $\gamma = 2\sqrt{mk}$ (critical damping) | $0 = \gamma < 2\sqrt{mk}$

Location of eigenvalues

Phase portrait



$E\lambda_1$ and $E\lambda_2$ collide