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Summary of class of 04/18/2017 MATH308

Phase plane (section 9.1)

$x' = Ax$, $x \in \mathbb{R}^2$, A is 2×2 matrix, $\det A \neq 0$.

The last condition implies that $Ax = 0 \iff x = 0$, i.e. the origin is a unique stationary (equilibrium point) of the system $x' = Ax$.

We want to know how the phase portrait (i.e. the collection of integral curves of the system) looks like in all possible case

Case 1 Distinct real eigenvalues

Assume that λ_1 and λ_2 are the eigenvalues of A, λ_1 & λ_2 are real, $\lambda_1 \neq \lambda_2$, and v^1, v^2 are the corresponding eigenvectors

Then the general solution is

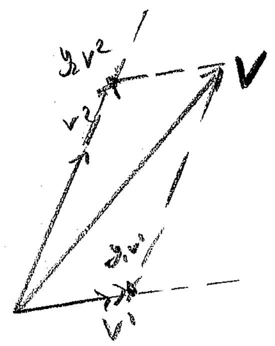
$$x(t) = c_1 e^{\lambda_1 t} v^1 + c_2 e^{\lambda_2 t} v^2 \qquad (1)$$

In order to understand how the curve given in the parametric form (1) looks in the the x_1x_2-plane let us study first how it looks like in the "distorted" coordinates (coordinate system) w.r.t. the basis $\{v^1, v^2\}$.

By the coordinates w.r.t the basis $\{v^1, v^2\}$ one means the following: any vector $v \in \mathbb{R}^2$ can be uniquely represented as

$$v = y_1 v^1 + y_2 v^2.$$

So, to any vector



v we can assign the pair of real numbers (y_1, y_2) that we call the coordinates of the vector v with respect to the basis $\{v^1, v^2\}$

For example, the cartesian coordinates (x_1, x_2) are coordinates with respect to the standard basis (e^1, e^2) where $e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The vector $x(t)$ from formula (1) has coordinate

$$\left(y_1(t), y_2(t)\right) = \left(C_1 e^{\lambda_1 t}, C_2 e^{\lambda_2 t}\right) \text{ or}$$

$$\begin{cases} y_1(t) = C_1 e^{\lambda_1 t} & (2) \\ y_2(t) = C_2 e^{\lambda_2 t} & (3) \end{cases} \rightarrow \text{parametric form of certain curve in } y_1 y_2\text{-plane}$$

What is this curve?

Assume that $C_1 \neq 0$ and express y_2 as a function of y_1 by first expressing t as a function of y_1 from eq. (2): $y_1 = C_1 e^{\lambda_1 t} \Rightarrow t = \frac{1}{\lambda_1} \ln \frac{y_1}{C_1}$ and then substituting this to (3) to obtain:

$$y_2 = C_2 e^{\frac{\lambda_2}{\lambda_1} \ln \frac{y_1}{C_1}} = C_2 \left(\frac{y_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

Denoting by $C = \frac{c_2}{c_1^{\lambda_2/\lambda_1}}$ and by $\boxed{\alpha = \frac{\lambda_2}{\lambda_1}}$

we get $y_2 = C y_1^{\alpha}$

$\boxed{\text{Case 1e: } \alpha > 0}$ → the case of **node**

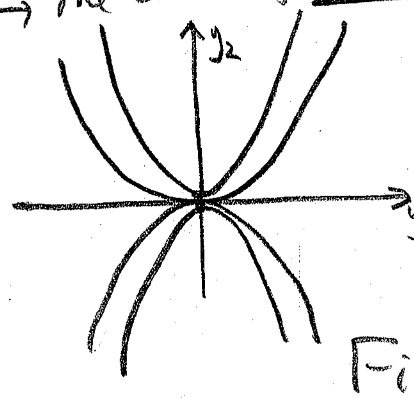


Fig 1

i.e. the eigenvalues are real and of the same sign. WLOG $|\lambda_2| > |\lambda_1|$

Then the curves $y_2 = C y_1^{\alpha}$ look like in Fig 1 i.e. (e.g. for $\alpha = 2$, they are parabolas with the vertices at the origin)

If $C_2 = 0$ then $y_2 = 0 \Rightarrow$ the curve is y_1-axis

If $C_1 = 0$ then $y_1 = 0 \Rightarrow$ the curve is y_2-axis

Now consider 2 subcases: (1ai) $\lambda_2 < \lambda_1 < 0$

In this case the direction of motion on each integral curve is toward the origin

In $y_1 y_2$-plane

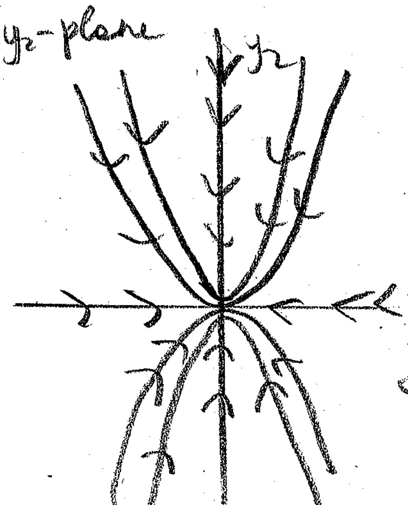


Fig 2

Returning to the original $x_1 x_2$-plane Fig 2 is distorted accordingly:

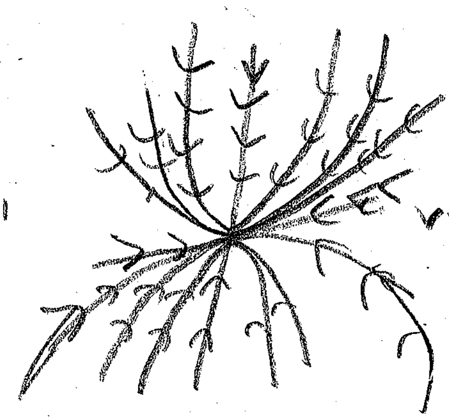


Fig 3

In this case the origin is called a nodal sink or stable node

(1a ii') $0 < \lambda_1 < \lambda_2$

In this case the direction of motion on each integral curve is <u>outward</u> the origin.

In $y_1 y_2$-plane

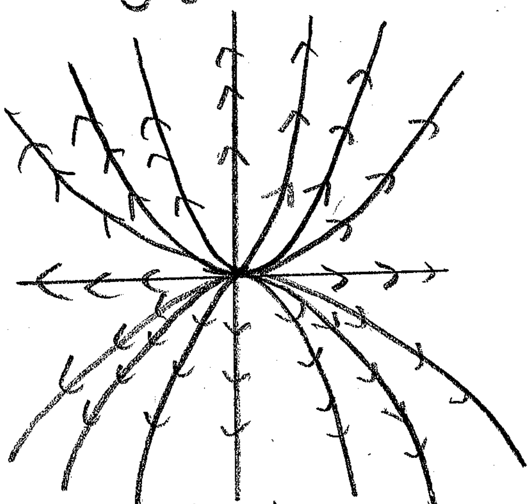


Fig. 4

In the original $x_1 x_2$-plane Fig 4 is distorted accordingly:

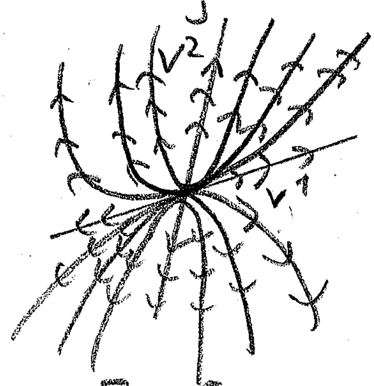


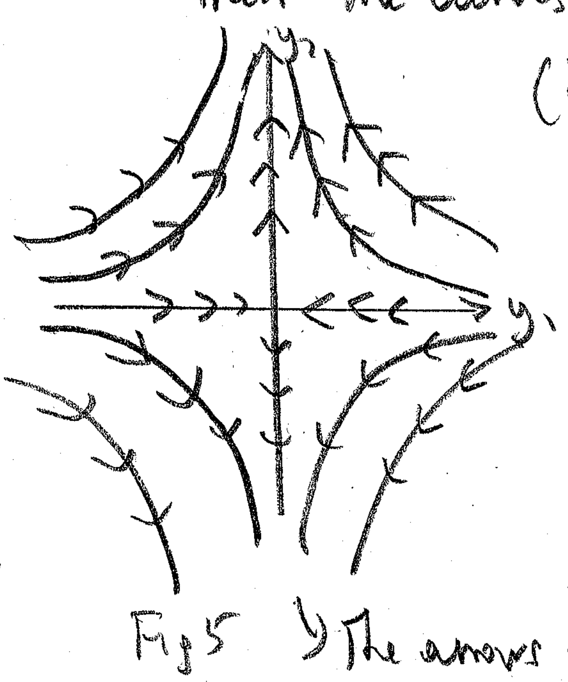
Fig 5

In this case the origin is called a <u>nodal source</u> or an <u>unstable node</u>

Case 1b $\boxed{d<0}$, i.e. the eigenvalue are <u>real</u> and <u>of the opposite signs</u> — $\boxed{\text{saddle}}$

WLOG $\lambda_1 < 0 < \lambda_2$

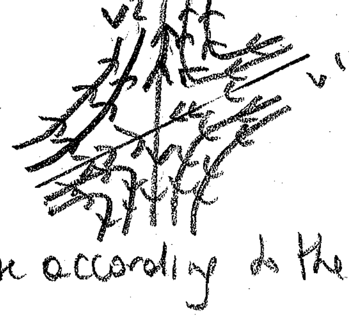
Then the curves $y_2 = C y_1^{d}$ look like in Fig. 6



(the axis are included as well as corresponding to $C_2 = 0$ or $C_1 = 0$)

(in the case $d = -1 \rightarrow$ hyperbolas)

In the original $x_1 x_2$-plane Fig 5 is distorted accordingly!



In this case the origin is called <u>saddle</u> point, it is <u>unstable</u>

Fig 5 ¹) The arrows are according to the condition $\lambda_1 < 0 < \lambda_2$

Also the eigenlines are called <u>separatrices</u>:

- The eigenline of the negative eigenvalue is called <u>stable separatrix</u>
- The eigenline of the positive eigenvalue is called <u>unstable separatrix</u>