Lecture notes of class of 04/25/2017 MATH 308

Nonlinear systems: stability and phase portrait (sections 9.2 & 9.3)

§1 Stability (from section 9.2)

Consider an arbitrary (not necessary linear) autonomous system of n first order differential equations

$$X' = F(x), \quad x \in \mathbb{R}^n \iff \begin{cases} x_1' = F_1(x_1, \dots, x_n) \\ x_2' = F_2(x_1, \dots, x_n) \\ \vdots \\ x_n' = F_n(x_1, \dots, x_n) \end{cases} \quad (1)$$

Def 1 A point $X^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ is called critical (also stationary, equillibrium) point of the system (1) if

$$F(x^0) = 0 \iff \begin{cases} F_1(x_1^0, \dots, x_n^0) = 0 \\ F_2(x_1^0, \dots, x_n^0) = 0 \\ \vdots \\ F_n(x_1^0, \dots, x_n^0) = 0 \end{cases}$$

Def 2 A critical point x^0 is called stable if for any neighborhood* U of x^0 there exists another (may be smaller) neighborhood V of x^0 s.t. for any solution $x(t)$ of (1) starting at V (i.e. such that $X(0) \in V$) $x(t)$ stay in U for $t > 0$ (see Fig 1)

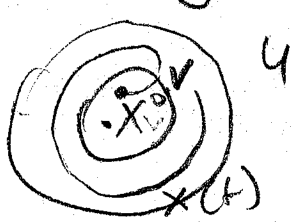

Fig 1

Def 3 A critical point x^0 is called unstable if it is not stable

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* By a neighbourhood of x^0 I mean here a ball around x^0 (a disk if $n=2$ or an interval if $n=1$)

Making the negation of Definition 2 one gets the following more detailed description of unstable critical point; A critical point x^0 of (1) is called unstable if there exists a neighborhood U of x^0 such that for any neighbourhood V of x^0 there exists a solution $x(t)$ starting at V (i.e. such that $x(0) \in V$) but going out of U for some $t > 0$.

<u>Def 4</u> A critical point x^0 is called <u>asymptotically stable</u> if it is stable and there exists a neighborhood V of x^0 s.t for any solution $x(t)$ starting at V

$$x(t) \xrightarrow[t \to +\infty]{} x^0.$$

Several remarks:

Remark 1) A center in linear planar system is an example of a <u>stable</u> critical point which is <u>not asymptotically stable</u> (see Fig 2):

It is <u>stable</u>, because:
For any neighb. U there exists an ellipse which is a solution of (1) and belongs to U, so we can take any neighb. of x_0 inside of this ellipse as the set V in Definition 2.

It is not asymptotically stable, because no solution $x(t)$ apart $x(t) \equiv 0$ approach the critical point 0.

Remark 2) In general the fact that there exists a neighborhood V of x^0 s.t. for any solution $x(t)$ starting at this neighborhood $x(t) \xrightarrow[t \to +\infty]{} x^0$ does not imply that x^0 is stable (see Fig 3)

A system on a sphere:

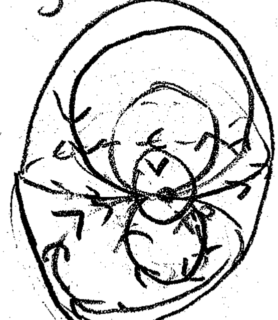


Fig 3

What all these definitions mean for linear systems?

(2) $\quad x' = Ax \quad \det A \neq 0 \Rightarrow x^0 = 0$ is the only critical point

From the form of the solutions of (2) we learned in chapter 7 (see also page 5 below) it follows that

1) 0 is asymptotically stable $\Leftrightarrow$ any eigenvalue λ of A satisfies: $\boxed{Re\, \lambda < 0}$

2) 0 is stable $\Rightarrow$ any eigenvalue of A satisfies $\boxed{Re\, \lambda \leq 0}$

2') In fact, x^0 is stable $\Leftrightarrow$ for any eigenvalue of λ, $Re\,\lambda \leq 0$ and the eigenvalues with $Re\,\lambda = 0$ have geometric multiplicity = algebraic multiplicity

3) If A has at least one eigenvalue λ' with $Re\,\lambda > 0$, then 0 is <u>unstable</u>;

3') In fact, 0 is unstable $\Leftrightarrow$ there exists an eigenvalue λ with $Re\,\lambda \geq 0$ and in case when $Re\,\lambda = 0$ its geometric multiplicity < algebraic multiplicity.

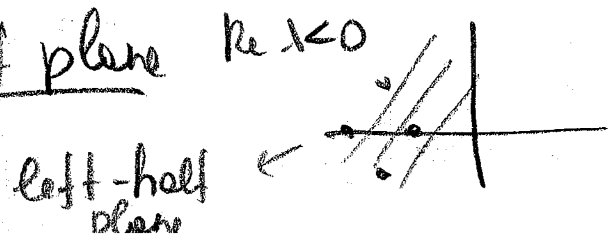
For <u>$n=2$</u> 2' & 3' are equivalent to the following more simply formulated conditions:

2") 0 is stable $\Leftrightarrow$ for any eigenvalue λ $Re\,\lambda \leq 0$

3") 0 is unstable $\Leftrightarrow$ for some eigenvalue λ $Re\,\lambda > 0$.

<u>Remark 3</u> The set of all eigenvalues of matrix A is called <u>the spectrum of matrix A</u>, denoted by $Spec\,A$

So, condition 1) for asymptotical stability of 0 is equivalent to the fact that $Spec\,A$ lies <u>entirely in the left-half plane</u> $Re\,\lambda < 0$



There are special condition when a polynomial has all roots in the left-half plane $\rightarrow$ <u>Routh-Hurwitz</u>

Also, in this case as a neighborhood V in Definition 4 one can take $V = \mathbb{R}^n$ i.e for any solution $x(t) \xrightarrow[t \to +\infty]{} 0$ → in this case 0 is said *globally asymptotically stable*

The reason for all this stability / nonstability conditions is that for linear systems each component of solutions is a linear combination of the functions of the form

$$e^{\mathrm{Re}\,\lambda t} \underbrace{P_k(t)}_{\text{polynomial}} \begin{pmatrix} \cos(\mathrm{Im}\,\lambda t) \\ \text{or} \\ \sin(\mathrm{Im}\,\lambda t) \end{pmatrix} \quad \text{or} \quad e^{\lambda t} P_k(t)$$

This is called quasi-polynomial

if λ is real. The degree of the polynomial $P_k(t)$ is 0 (i.e. $P(t)$ is just a constant), if geometric multiplicity of λ = algebraic multiplicity of λ and it is > 0 otherwise, so for example if $\mathrm{Re}\,\lambda < 0$ then

$$e^{\mathrm{Re}\,\lambda t} P(t) \begin{matrix} \cos(\mathrm{Im}\,\lambda t) \\ \sin(\mathrm{Im}\,\lambda t) \end{matrix} \xrightarrow[t \to +\infty]{} 0$$

and if $\mathrm{Re}\,\lambda > 0$ or $\mathrm{Re}\,\lambda = 0$ but degree of P is > 0 then $e^{\mathrm{Re}\,\lambda t} P(t) \begin{matrix} \cos(\mathrm{Im}\,\lambda t) \\ \sin(\mathrm{Im}\,\lambda t) \end{matrix}$ is unbounded

Nonlinear (locally linear) systems

Linearization principle (section 9.3 with an example from section 9.4)

Let $n=2$. So our system (1) consists of 2 equations

$$x' = f(x,y) \qquad (3)$$
$$y' = g(x,y)$$

(the case $n=2$ is considered for simplicity, all theorems below are true for arbitrary n)

Assume that $= (x_0,y_0)$ is a stationary point of (3) i.e $\begin{cases} f(x_0,y_0)=0 \\ g(x_0,y_0)=0 \end{cases}$

Expand f and g into the Taylor expansions up to linear terms:

$$f(x,y) = \overset{=0}{f(x_0,y_0)} + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) +$$
$+$ a remainder (of order higher than linear in $x-x_0$ and $y-y_0$)

$$g(x,y) = \underset{\shortparallel\atop 0}{g(x_0,y_0)} + \frac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0) +$$
$+$ a remainder (of order higher than linear in $x-x_0$ and $y-y_0$)

Substituting these expansions into (3) and ignoring the remainders (as terms smaller than the linear terms for sufficiently small $x-x_0$ and $y-y_0$ we get

$$x' = \frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$y' = \frac{\partial g}{\partial x}(x_0,y_0)\,(y-y_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

Make a substitution $u = x - x_0 \Rightarrow u' = x'$
$v = y - y_0 \Rightarrow v' = y'$

So
$$\left.\begin{aligned} u' &= \frac{\partial f}{\partial x}(x_0,y_0)\,u + \frac{\partial f}{\partial y}(x_0,y_0)\,v \\ v' &= \frac{\partial g}{\partial x}(x_0,y_0)\,u + \frac{\partial g}{\partial y}(x_0,y_0)\,v \end{aligned}\right\} \quad (4)$$

$\swarrow$

a linear system with matrix

$$J(x_0,y_0) = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}(x_0,y_0)}_{}$$

<u>Jacobi matrix</u> of the system at (x_0,y_0)

<u>Def 5</u> System (4) is called the <u>linearization</u> of system (3) at the critical point (x_0,y_0)

The induction is that since we ignore very small terms when passing from (3) to (4)

The behavior of solutions of the original nonlinear system near (x_0, y_0) will be similar to the behavior of solutions of its linearization (4) and the latter we know from what we studied in chapter 7. However, we need an extra assumption on the eigenvalues of $J(x_0, y_0)$

Theorem 1 If there is no eigenvalues of $J(x_0, y_0)$ lying on the imaginary axis (i.e. for any eigenvalue λ of $J(x_0, y_0)$ $Re\,\lambda \neq 0$) then the stability properties of the stationary point (x_0, y_0) of the original system (3) are the same as the stability properties of the origin for its linearization (4) at (x_0, y_0).

Also, the phase portrait looks similar.
For example, if $J(x_0, y_0)$ has two real eigenvalues λ_1 and λ_2 of opposite signs, $\lambda_1 < 0$ and $\lambda_2 > 0$ then there exist 2 curves Γ_- and Γ_+ in a neighb U of (x_0, y_0) passing through (x_0, y_0) and such that if $(x(0), y(0)) \in \Gamma_+$ then $(x(t), y(t)) \xrightarrow[t \to +\infty]{} (x_0, y_0)$

and if $(x(0), y(0)) \in \Gamma_+$ then $(x(t), y(t)) \xrightarrow[t \to -\infty]{} (x_0, y_0)$

In addition the tangent line to Γ_- at (x_0, y_0) coinsides with the eigenline of $\lambda_1 < 0$ of J and the tangent line of Γ_+ at (x_0, y_0) coinsides with the eigenline of $\lambda_2 > 0$ of J. The curves Γ_- and Γ_+ are called the stable and unstable separatrices of (x_0, y_0), respectively. The phase portrait of the original system (3) near (x_0, y_0) is a "distortion" of the phase portrait of its linearization (see Figs 4&5)

Phase portrait of the linearization

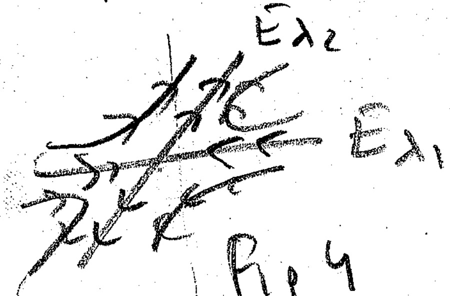
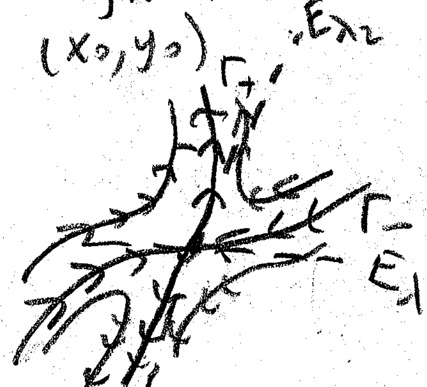


Fig 4

Phase portrait of the original system near (x_0, y_0)



Why the case of $\text{Re}\,\lambda = 0$ is excluded from Theorem 1? This case is sensitive to nonlinear perturbation as a threshold between stability and instability, so in this case stability cannot be decided by linearization as the following example shows:

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + \varepsilon x_2^3 \end{cases} \qquad (5)$$

For $\varepsilon = 0 \to$ just a harmonic oscillator $\to$ linear system with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and eigenvalues $\pm i$ on the imaginary axis.

The function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ is in fact a total energy of harmonic oscillator. Let $x_t(t) = (x_1(t), x_2(t))$ be a solution of (5)

Calculate $\frac{d}{dt} V(x_1(t), x_2(t))$, i.e. the rate of change of the total energy along the solution ($\Leftrightarrow$ the rate of change of the square of the distance to the origin)

$$\frac{d}{dt} V(x_1(t), x_2(t)) = x_1 \underbrace{x_1'}_{x_2} + x_2 \underbrace{x_2'}_{(-x_1 + \varepsilon x_2^3)} =$$

$$= \underbrace{x_1 x_2 - x_2 x_1}_{=0} + \varepsilon x_2^4 = \varepsilon x_2^4$$

(the conservation of energy for the harmonic oscillator)

So if $\varepsilon < 0$ then $\frac{d}{dt} V(x_1(t), x_2(t)) \leq 0$

(and equal to 0 only if $x_2(t) = 0$) $\Longrightarrow$

stability (and actually asymptotic stability)

The function V is an example of the <u>Lyapunov</u>

<u>function</u> (see section 9.6)

If $\varepsilon < 0$ then $\frac{d}{dt} V(x_1(t), x_2(t)) \geq 0 \Rightarrow$

(and equal to 0 only if $x_2(t) = 0$) $\Rightarrow$ <u>unstability</u>

As this example shows the stability / unstability

is decided by the nonlinear term.

<u>Remark 4</u> Note that we actually proved

Theorem 1 in the case $n = 1$ (phase line)

[see notes 10 of week 4, item 6.1 and 6.2

on page 4, discussions of the sign of the derivatives]

In this case the original system is $y' = f(y)$

and the linearization is $u' = f'(y_0) u$, so

$J = f'(y_0) \Rightarrow f'(y_0)$ is an eigenvalue of J.

 Example (related to the model of competing species, section 9.4)

Consider the following system in $\mathbb{R}^2$:

$$\begin{cases} x' = x(7-x-2y) \\ y' = y(5-y-x) \end{cases} \Leftrightarrow \begin{array}{l} x' = 7x - x^2 - 2xy \\ y' = 5y - y^2 - xy \end{array}$$

x & y denotes a population of 2 species

Terms $7x$ & $5y$ represent the natural grows of the first and second species population without competion, the term $-x^2$ and $-y^2$ represent the competition within each species and the mixed terms $-2xy$ and $-xy$ represent the effect of the competition of different species on the population of the first one and the second one, respectively.

a) Find all critical points.

For this we have to solve the nonlinear algebraic system of equations:

$$\begin{cases} x(7-x-2y)=0 & (6.1) \\ y(5-y-x)=0 & (6.2) \end{cases}$$

(6.1) $\Leftrightarrow$ either $x=0$ or $7-x-2y=0$

(6.2) $\Leftrightarrow$ either $y=0$ or $5-y-x=0$

We have 4 possibilities

1) $x=0$ & $y=0$ $\Rightarrow (0,0)$ is a critical point
2) $x=0$ & $5-y-x=0 \Rightarrow 5-y=0 \Rightarrow y=5 \Rightarrow (0,5)$ is a critical point

3) $7-x-2y=0$ & $y=0 \Rightarrow 7-x=0 \Rightarrow y=7 \Rightarrow (7,0)$ is a critical point

4) $\begin{cases} 7-x-2y=0 \\ 5-y-x=0 \end{cases} \Leftrightarrow \begin{cases} x+2y=7 \\ x+y=5 \end{cases} \Rightarrow y=2 \Rightarrow x=3 \Rightarrow$ $(3,2)$ is a critical point

So we have 4 critical points $(0,0), (0,5), (7,0), (3,2)$

B) For each critical point find the corresponding linearization and classify the critical points based on this and determine their stability properties
(by classification are means determing the type of the critical point for the linearization and by stability properties one means to say whether is it stable, asymptotically stable, or unstable)
First find the Jacobi matrix $J(x,y)$

In our case

$$f(x,y) = x(7-x-2y) = 7x - x^2 - 2xy$$

$$g(x,y) = y(5-y-x) = 5y - y^2 - xy$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 7-2x-2y & -2x \\ -y & 5-2y-x \end{pmatrix}$$

Hence

1) For $(x_0, y_0) = (0,0)$

$$J(0,0) = \begin{pmatrix} 7 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow$$ The linearization is $\begin{cases} u' = 7u \\ v' = 5v \end{cases}$

$J(0,0)$ is diagonal $\Rightarrow$ its eigenvalues are $7 \& 5 \Rightarrow$ positive real, distinct $\Rightarrow$ $\boxed{\text{nodal source \& unstable}}$

2) For $(x_0, y_0) = (0,5)$

$$J(0,5) = \begin{pmatrix} 7-2\cdot 5 & 0 \\ -5 & 5-2\cdot 5 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -5 & -5 \end{pmatrix} \Rightarrow \begin{cases} u' = -3u \\ v' = -5u - 5v \end{cases}$$

$J(0,5)$ is lower triangular $\Rightarrow$ its eigenvalues are $-3 \& -5 \Rightarrow$ negative real, distinct $\Rightarrow$ $\boxed{\text{nodal sink, asymptotically stable}}$

3) $(x_0, y_0) = (7, 0)$

$$J(7,0) = \begin{pmatrix} 7-2\cdot 7 & -14 \\ 0 & 5-7 \end{pmatrix} = \begin{pmatrix} -7 & -14 \\ 0 & -2 \end{pmatrix} = \begin{cases} u' = -7u - 14v \\ v' = -2v \end{cases}$$

$J(7,0)$ is upper triangular $\Rightarrow$ the eigenvalues are -7 & -2

negative real, distinct $\Rightarrow$ $\boxed{\text{nodal sink}}$ $\boxed{\text{asymptotically stable}}$

4) $(x_0, y_0) = (3, 2) \Rightarrow$

$$J(3,2) = \begin{pmatrix} 7-6-4 & -6 \\ -2 & 5-2\cdot 2-3 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ -2 & -2 \end{pmatrix}$$

$\det(J - \lambda I) = \lambda^2 - \operatorname{tr} J \lambda + \det J = \lambda^2 + 5\lambda - 6 \Rightarrow$

$D = 25 + 24 = 49$

$\lambda_1 = \frac{-5+7}{2} = 1$

$\lambda_2 = \frac{-5-7}{2} = -6$

λ_1 & λ_2 are real of opposite signs $\rightarrow$ $\boxed{\text{saddle point}}$ $\boxed{\text{unstable}}$

Important remark For $n=2$ if $\det J < 0$ then

it is saddle : in this case 1) the discriminant is positive

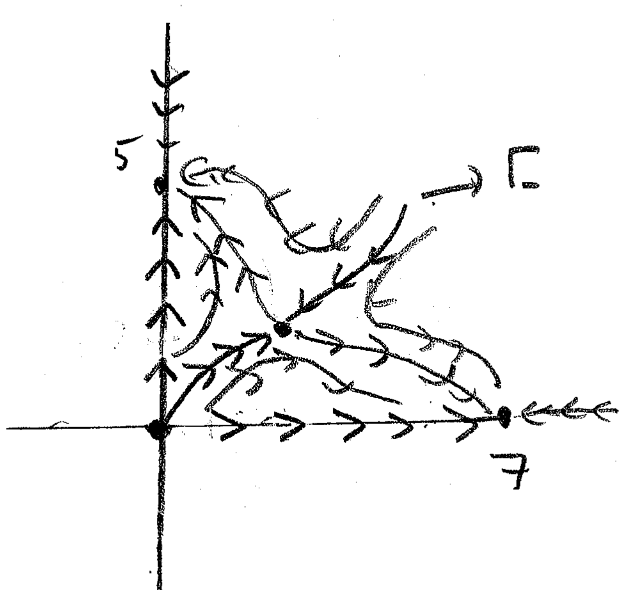
$(D = (\operatorname{tr} J)^2 - 4 \det J > 0) \Rightarrow \lambda_1$ & λ_2 are real

2) $\lambda_1 \lambda_2 = \det J < 0 \Rightarrow \lambda_1$ & λ_2 are of opposite signs

So you do not actually need to find λ_1 & λ_2 to make the conclusion about the saddle in this case.

c) Sketch the phase portrait in the first quadrant (taking into account that the system describes the competing species model, the first quadrant is the only set when the behavior of the system is interesting)

To make a sketch we first do it near each critical point based on Theorem 1 and our calculations in item b) and then glue all these sketches together:



d) Make conclusion on coexistence and, if no coexistence, then which species will win (depending on the initial conditions).

The only critical point with both components being positive is $(3,2)$, but it is a saddle point $\Rightarrow$ unstable $\Rightarrow$ $\boxed{\text{no coexistence in general}}$:

- The solution will approach $(3,2)$ as $t \to +\infty$ if and only if the initial conditions are on the stable separatrix Γ_- $\Rightarrow$ the coexistence occurs only in this situation but this situation is sensitive to small perturbation of initial conditions:
- If the initial conditions are above Γ_-, then the solution will approach the point $(0,5)$, so the second species will win and the first one will die out.
- If the initial conditions are below Γ_-, then the solution will approach the point $(7,0)$, so the first species will win and the second one will be extincted.

Page 18) Discussion of general competing species model (section 9.4)

$$\begin{cases} x' = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y' = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases}$$

All constants are positive

Assumption: The system

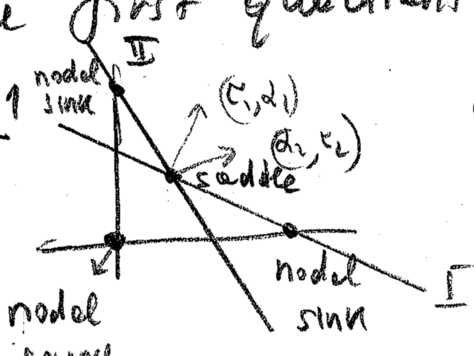
$$\begin{cases} \sigma_1 x + \alpha_1 y = \varepsilon_1 \\ \alpha_2 x + \sigma_2 y = \varepsilon_2 \end{cases}$$

has one solution in the first quadrant (then this solution is a candidate to the limiting point for coexistence)

Denote by I the line $\sigma_1 x + \alpha_1 y = \varepsilon_1$ (called x-nullclines) and by II the line $\alpha_2 x + \sigma_2 y = \varepsilon_2$ (called y-nullclines)

Our assumption means that the lines I & II intersect in the first quadrant. There are two cases:

Case 1



$\iff$ $\boxed{\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0}$

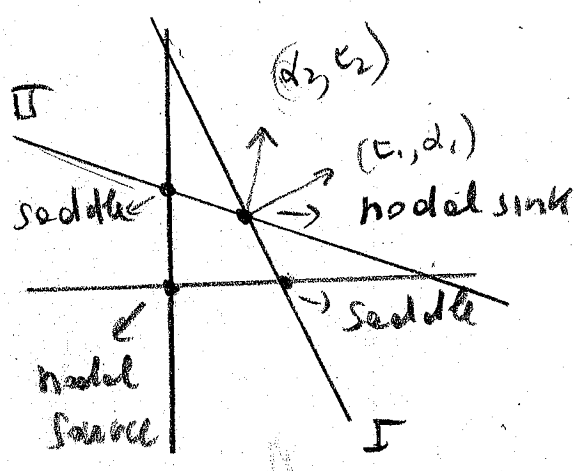
(because the shortest rotation from (σ_1, α_1) to (α_2, σ_2) is clockwise $\iff \begin{vmatrix} \sigma_1 & \alpha_1 \\ \alpha_2 & \sigma_2 \end{vmatrix} < 0 \iff \sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$)

This was in our previous example

$r_1 = 1 \quad \alpha_1 = 2 \quad \Rightarrow \quad r_1 r_2 - \alpha_1 \alpha_2 = 1 - 2 = -1 < 0$

$\alpha_2 = 1 \quad r_2 = 1$

Case 2

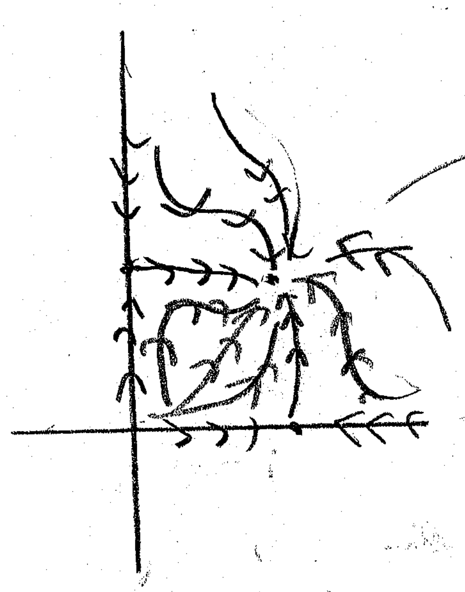


$\Leftrightarrow \quad r_1 r_2 - \alpha_1 \alpha_2 > 0$

(because the shortest rotation from (r_1, α_1) to (α_2, r_2) is counterclockwise $\Leftrightarrow$

$\begin{vmatrix} r_1 & \alpha_1 \\ \alpha_2 & r_2 \end{vmatrix} > 0 \Leftrightarrow r_1 r_2 - \alpha_1 \alpha_2 > 0$)

The phase portrait in this case is



In this case there is coexistence!! ☺

the forth critical point (i.e. with both components being positive) is asymptotically stable (nodal sink)