3. Examples of equations that are initially not separable but can be reduced to separable.

1. **Type 1** Homogeneous (nonlinear) equation (do not confuse with linear homogeneous equation discussed in the first week and that will be discussed later) are equations of the type

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \tag{1}$$

where F is a given function of a single variable. In other words, the right-hand side of the equation depends on the ratio $\frac{y}{x}$ only.

Make a substitution

$$u(x) = \frac{y(x)}{x}$$

.

Then $y(x) = xu(x)$. Substitute into equation (1):

Left-hand side:

$y' =$

Right-hand side

$F(y/x) =$

Then we arrive to the equation

$$xu'(x) + u = F(u) \quad \Leftrightarrow \quad xu'(x) = F(u) - u$$

which is separable.

2.

EXAMPLE 1. *Reduce the equation* $\frac{dy}{dx} = \frac{3x+2y}{2x-3y}$ *to a separable equation*

3. **Type 2** Equations of the type

$$\frac{dy}{dx} = F(ax + by + c), \tag{2}$$

where F is a given function of a single variable and a, b, c are constants, $b \neq 0$.

Make a substitution

$$u(x) = ax + by(x) + c$$

.

Then $y(x) = \frac{1}{b}(u(x) - ax - c)$. Substitute into equation (2):
Left-hand side:
$y' =$
Right-hand side
$F(ax + by + c) =$

Then we arrive to the equation

$$\frac{1}{b}(u' - a) = F(u) \quad \Leftrightarrow \quad u' = bf(u) + a$$

which is separable.

EXAMPLE 2. *Reduce the equation* $\frac{dy}{dx} = \frac{1}{2y+x}$ *to a separable equation*

4. Linear Equations: Method of Integrating Factor (2.1)

1. A first order ODE is called **linear** if it is expressible in the form

$$y' + p(t)y = g(t) \tag{3}$$

   where $p(t)$ and $g(t)$ are given functions.

2. If $g(t) \equiv 0$ then

$$y' + p(t)y = 0 \tag{4}$$

   is called a **homogeneous** liner ODE. Otherwise (3) is called a **non-homogeneous** liner ODE.

3. The equation (3) is separable if and only if there is a real constant α such that $g(t) = \alpha p(t)$. (We already know how to solve it!)

4. The method to solve (3) for arbitrary $p(t)$ and $q(t)$ is called 2in

The Method of Integrating Factors

Step 1 Put ODE in the form (3).

Step 2 Find the integrating factor μ by solving the equation $\mu' = p(t)\mu$, i.e,

$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

Note: Any μ, satisfying $\mu' = p(t)\mu$, will suffice here, thus one can take the constant of integration $C = 0$ when calculating $\int p(t)\mathrm{d}t$.

Step 3 Multiply both sides of (3) by μ and use the Product Rule for the left side to express the result as

$$(\mu(t)y(t))' = \mu g(x) \tag{5}$$

Step 4 Integrate both sides of (5). Note: Be sure to include the constant of integration in this step to get all solutions!

Step 5 Solve for the solution $y(t)$. In this step make sure to divide all terms by μ, including the term containing the constant of integration of the previous step.

5. Consider

$$y' - 3xy = -xe^{x^2}.$$

(a) Find the general solution.

(b) Find the solution satisfying the initial condition $y(0) = y_0$.

(c) How do the solutions behave as x becomes large (i.e $x \to +\infty$)? Does that behavior depend on the choice of the initial value y_0?

5. Models of mixing (from section 2.3) as an example of linear nonhomogeneous ODE

(a) **Mixing Problems** serve as a model for problem of discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, for example.

(b) *Chemicals in Tank, or Mixing.* Assume that a tank contains V gal of water and Q_0 lb of some substance (salt, or chemical, for example) is dissolved in it. Suppose that the water containing $\gamma(t)$ lb/gal of substance is entering the tank with the rate r gal/min (the concentration of the chemicals in the entering water varies in time) and then well stirred mixture is draining from the tank at the same rate. Find the amount $Q(t)$ of the substance in tank at any time (Do not solve, just find IVP for $Q(t)$).

(c) *Chemicals in Pond* (from the textbook) A pond contains 10 million gal of fresh water. Stream water containing an undesirable chemical flows into the pond at the rate 5 million gal/yr, and the mixture in the pond flows out through an overflow culvert at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time t, measured in years, according to the expression $\gamma(t) = 2 + \sin(2t)$g/gal.

i. Construct a mathematical model of this flow process.

ii. Determine the amount of chemical in the pond at any time.

iii. Plot the solution and describe in words the effect of the variation of the incoming concentration.