

6. Exact Equations (section 2.6)

1. Consider a first order differential equation

$$P(x, y) + Q(x, y)y' = 0 \quad (\text{or } P(x, y)dx + Q(x, y)dy = 0) \quad (1)$$

with P and Q having continuous partial derivatives in the region R of \mathbb{R}^2 .

DEFINITION 1. *The equation (1) is called exact on the region R if there exists a differentiable function Φ on the region \mathbb{R} such that*

$$\begin{cases} \Phi_x(x, y) = P(x, y) \\ \Phi_y(x, y) = Q(x, y) \end{cases} \quad (2)$$

for every $(x, y) \in R$.

REMARK 2. *Note that exact equations are very special: system (2) is the system of two equations for one unknown function Φ and it does not have a solution for an arbitrary choice of the pair of functions (P, Q) . In order that the system (2) will have a solution the pair (P, Q) must satisfy a certain compatibility condition (see equation (3) below)*

2. Assume that Φ satisfies the system (2). Then $y(x)$ is a solution of the equation (1) if and only if

$$\frac{d}{dx}\Phi(x, y(x)) =$$

In other words, an exact equation (1) represents the **exact differential** of the function $\Phi(x, y)$, i.e $d\Phi = 0$ (this is the origin of the word exact).

Therefore,

$$\Phi(x, y(x)) \equiv C$$

for some constant C .

CONCLUSION *Any integral curve $y = y(x)$ of an exact equation (1) lies on a level curve of the function $\Phi(x, y)$ and the general solution of equation (1) is given in an implicate form by*

$$\boxed{\Phi(x, y) = C}$$

3. *EFFECTIVE TEST for Exactness:* If ODE (1) is exact on the region R , then

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \quad \text{on } R \quad (3)$$

Proof.

□

PROPOSITION 3. *If the region R is simply connected (i.e. without holes, for example $R = \mathbb{R}^2$) then ODE (1) is exact on R if and only if the relation (3) holds on R .*

4. *Determine whether the following ODE are exact:*

(a) $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$

(b) $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$

REMARK 4. If you are asked to verify exactness you are not allowed to multiply (or divide) you equation by a nonconstant function: such operation will destroy the exactness.

5. *Relation to the topic of “Conservative Vector Fields” from Calculus-3 (Stewart, Section 14.3)*
Consider the vector field $\vec{F} = \langle P, Q \rangle$ in the region R . The system (2) is equivalent to the equation

$$\text{grad } \Phi = \vec{F}. \quad (4)$$

In other words, vector fields \vec{F} is conservative, the function Φ is a potential of \vec{F} and (3) is the test to check whether \vec{F} is conservative in a simply connected region R .

CONCLUSION: *A general solution to an exact differential equation can be found by the method used in Calculus 3 to find a potential function for a conservative vector field in a plane region.*

6. Solve $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$.