7. Nonexact equation that can be made exact using integrating factors depending on single variable (section 2.6)

1. Consider again a first order differential equation

$$P(x,y) + Q(x,y)y' = 0 \quad (\text{or} \quad P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = 0) \tag{1}$$

on a simply connected region R of $\mathbb{R}^2$. and assume that it is not exact, i.e. equivalently $P_y \neq Q_x$ on R.

Goal To find a function μ such that after multiplication of (1) by μ the equation becomes exact. Such function μ is called an *integrating factor for equation* (1). For this we need that

$$(\mu P)_y = (\mu Q)_x$$

Using the product rule we get:

$$P\mu_y - Q\mu_x + (P_y - Q_x)\mu = 0. \tag{2}$$

This is a first order linear partial differential equation (PDE) for the function μ and to solve it is equaly hard as to solve the original equation (1). So, in general, the idea of making equation (1) exact does not give an efficient method to solve it.

However, in some specific cases, this idea works perfectly.

For example, suppose we can find the integrating factor which is a function of x alone

QUESTION: Under what condition on P and Q is it possible?

ANSWER: Integrating factor $\mu = \mu(x)$ (i.e depending on x alone) exists if and only if $\dfrac{P_y - Q_x}{Q}$ is a function of x alone. In this case this integrating factor satisfies the equation

$$\frac{d\mu}{dx} = \frac{P_y - Q_x}{Q}\mu$$

Similarly, integrating factor $\mu = \mu(y)$ (i.e depending on y alone) exists if and only if $\dfrac{P_y - Q_x}{P}$ is a function of y alone. In this case this integrating factor satisfies the equation

$$\frac{d\mu}{dy} = \frac{Q_x - P_y}{P}\mu$$

2. Solve $(3x^2y + 2xy + y^3)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0$

3. Relation to separable equations.

4. **Relation to the method of integrating factor for linear equations from section 2.1** Let us show that the method of integrating factor for linear equations from section 2.1 is a particular case of the method discussed here.