9. Existence and Uniqueness of Solutions, Differences between Linear and Nonlinear Equations (section 2.4)

In general one cannot solve explicitly the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1}$$

The main questions here: *Does a solution of* (1) *exist? Is it unique?*

1. **Existence via the Euler method (brief)** Existence can be established under the assumption that the right-hand side $f(t, y)$ of 1 is continuous using approximation by **Euler's broken lines:** (very briefly, for more detail see sections 2.7 and 8.1, but this is not required):

   From continuity of $f(t, y)$ it will follow that the Euler broken lines will converge to the actual solution as step sizes go to 0.

2. However, continuity of $f(t, y)$ does not guarantee the uniqueness of solutions of IVP (1)

   EXAMPLE 1. $y' = y^{2/3}, \quad y(0) = 0.$

   The idea behind the construction of such examples: there are two different curves on a plane passing through a point and tangent to each other at this point.

   So, we need additional assumptions on $f(t, y)$ (see Theorem 3 below).

3. Another complication is that even if $f(t, y)$ is continuous (which implies the existence of the solution of IVP (1)) it is not guaranteed that the solution is defined for all time moments (solutions may go to infinity at final time, i.e. explode or **blow up**):

   EXAMPLE 2. $y' = y^2, \quad y(0) = y_0 > 0.$

   That is why the formulation of the existence and uniqueness theorem looks a bit cumbersome.

4. **Existence and Uniqueness of Solutions**

   THEOREM 3. *[Theorem 2.4.2 of the book] Let the functions f and $\partial f/\partial y$ be continuous in some rectangle*

   $$R = \{(t, y) | \alpha < t < \beta, \quad \gamma < y < \delta\}$$

   containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $I = \{t| \ \alpha < t < \beta\}$, there is a **unique** *solution $y = y(t)$ of the initial value problem*

   $$y' = f(t, y), \quad y(t_0) = y_0.$$

5. Verify that the equation in Example 1 does not satisfy the conditions of Theorem 3.

6. Does the equation in Example 2 satisfy the conditions of Theorem 3?

   So, again, this shows that assumptions in Theorem 3 guarantee the existence of solution only for values of t which are sufficiently closed to t_0, but not for all t

7. Geometric consequence of the uniqueness is that two different integral curves never intersect each other.

8. **Existence and Uniqueness of Solutions of <u>Linear</u> ODE**

   THEOREM 4. *[Theorem 2.4.1 of the book] If the functions $p(t)$ and $g(t)$ are continuous on the interval $I = \{t|\ \alpha < t < \beta\}$, then for ant $t = t_0$ on I, there is a* **unique** *solution $y = y(t)$ of the initial value problem*

$$y' + p(t)y = g(t), \quad y(t_0) = y_0. \tag{2}$$

9. Note that the conditions of the Theorem 1 hold automatically for linear ODE. So, Theorem 4 gives stronger conclusion for linear equations.

10. Determine (without solving the problem) an interval in which the solution of the given IVP is certain to exist:

$$(t-3)y' + (\ln|t|)y = 2t$$

   (a) $y(1) = 2$;

   (b) $y(5) = 7$;

   (c) $y(-1) = 10$.

11. Consider

$$ty' + 2y = 4t^2 \tag{3}$$

   (a) Determine (without solving the problem) an interval in which the solution (3) satisfying $y(t_0) = y_0$ with $t_0 > 0$ is certain to exist.

   (b) The solution of IVP from item (a) found by the method of integrating factor is given:

$$y(t) = t^2 + \frac{C}{t^2}, \qquad C = t_0^2(y_0 - t_0^2).$$

   Using that information discuss the domain of the solution and compare your conclusion with the answer of (a).

   Conclusion from this example: The domain of the solution given by Theorem 4 might be smaller than the actual domain.