# 11. Solutions of linear homogeneous equations of second order. The Wronskian (section 3.2).

1. **Second order ODE**:
$$y'' = f(t, y, y')$$
where $f$ is a function whose domain is in $\mathbb{R}^3$. A solution of such an equation is a function $y = y(t)$, twice differentiable in some interval $I$, s.t. for all $t$ in $I$
$$y''(t) = f(t, y(t), y'(t)).$$

2. By Newton's Second Law $F = ma$. We know that acceleration $a(t)$ is the second derivative of position $x(t)$, which yields
$$x'' = \frac{1}{m}F(x, x').$$

3. Find general solution of $y'' = 0$.

4. Reduction to the system of first order ODEs by a substitution:

   - Particular case $y'' = f(t, y')$ use the substitution $u = y'$.

   - General case $y'' = f(t, y, y')$ use the substitution $x_1 = y, \quad x_2 = x_1'$.

5. By analogy with the existence and uniqueness theorem for a single first order ODE we have

   THEOREM 1. *If* $f$*,* $\dfrac{\partial f}{\partial x_1}$ *and,* $\dfrac{\partial f}{\partial x_2}$ *are continuous in a region*

   $$R = \{\alpha < t < \beta, \quad \alpha_1 < x_1 < \beta_1, \quad \alpha_2 < x_2 < \beta_2\}$$

   *then there exists a unique solution through a point* $(t_0, y_0, v_0)$ *in* $R$ *(equivalently, there is an interval* $t_0 - h < t < t_0 + h$ *in which there exists a unique solution of the IVP* $y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = v_0$*.)*

6. Second order **linear** ODE:

   $$y'' + p(t)y' + q(t)y = g(t)$$

   or

   $$y'' = -p(t)y' - q(t)y + g(t)$$

   where $p, q$, and $g$ are continuous on an interval $I = (\alpha, \beta)$

7. By analogy with the existence and uniqueness theorem for a linear first order ODE we have

   THEOREM 2. *If the functions* $p(t), q(t)$ *and* $g(t)$ *are continuous on the interval then for any* $t = t_0$ *on* $I$*, there is a* **unique** *solution* $y = y(t)$ *of the IVP*

   $$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0. \tag{1}$$

## Linear HOMOGENEOUS ODE of second order

8. Question: Can the function $y = \sin(t^2)$ be a solution on the interval $(-1, 1)$ of a second order linear homogeneous equation with continuous coefficients?

9. Consider a linear homogeneous ODE

   $$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

   with coefficients $p$ and $q$ continuous in an interval $I$.

10. **Superposition Principle**

    - Sum $y_1(t) + y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.

*Proof.*

□

- A scalar multiple $Cy(t)$ of any solution $y(t)$ of (2) is itself a solution.

  *Proof.*

  □

COROLLARY 3. *Any* `linear combination` $C_1y_1(t) + C_2y_2(t)$ *of any two solutions* $y_1(t)$ *and* $y_2(t)$ *of (2) is itself a solution.*

*Proof.*

□

11. `Why Superposition Principle is important?` Once two (linearly independent) solutions of a linear homogeneous equation are known, a whole class of solutions is generated by linear combinations of these two.

12. **WRONSKIAN** of the functions $y_1(t)$ and $y_2(t)$:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

13. *What does Wronskian measure?* Note that

$$\frac{d}{dt}\left(\frac{y_2(t)}{y_1(t)}\right) = \frac{y_1(t)y_2'(t) - y_2(t)y_1'(t)}{y_1^2(t)} = \frac{W(y_1, y_2)(t)}{y_1^2(t).}$$

This shows that $y_2(t) = Cy_1(t)$ *on an interval* $I$ *for some constant* $C$ *if and only if* $W(y_1, y_2)(t) \equiv 0$ (more precisely, for the if part you need additional assumptions, for example, that $\frac{y_1(t)}{y_2(t)}$ is differentiable on $I$).

Indeed,

14. Suppose that $y_1(t)$ and $y_2(t)$ are two differentiable solutions of (2) in the interval $I$ such that $W(y_1, y_2)(t) \neq 0$ somewhere in $I$, then every solution is a linear combination of $y_1(t)$ and $y_2(t)$.

In other words, the family of solutions $y(t) = C_1y_1(t) + C_2y_2(t)$ with arbitrary coefficients $C_1$ and $C_2$ includes every solution of (2) if and only if there is a points $t_0$ where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental set** of solutions of (2).

*Proof.*

□

15.

REMARK 4. Wronskian $W(y_1, y_2)(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) either is zero for all $t$ or else is never zero.

*Explanation*

16. Another explanation of the same remark is via the following theorem Abel's Theorem (Theorem 3.2.7 of the book):

THEOREM 5. *( Abel's Theorem, see Theorem 3.2.7 of the book)If the functions $y_1$ and $y_2$ are solutions of the second order linear homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0,$$

*then the Wronskian $W(t) = W(y_1, y_2)(t)$ of these functions satisfies the first order linear homogeneous differential equation*

$$W' + p(t)W = 0.$$

17. Confirm that $\sin t$ and $\cos t$ are solutions of $y'' + y = 0$. Then solve the IVP

$$y'' + y = 0, \quad y(\pi) = 0, \quad y'(\pi) = -5$$

## Appendix: Facts from Algebra

1. 
   - FACT 1: **Cramer's Rule** for solving the system of equations

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

   The rule says is that if the `determinant of the coefficient matrix is not zero`, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

   then the system has a unique solution $(x, y)$ given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

   - FACT 2: If determinant of the coefficient matrix is zero then either there is no solution, or there are infinitely many solutions.
   - FACT 3. The homogeneous system of linear equations

$$\begin{aligned} a_1 x + b_1 y &= 0 \\ a_1 x + b_2 y &= 0 \end{aligned}$$

   always has the "trivial" solution $(x, y) = (0, 0)$. By Cramer's rule this is the only solution if the determinant of the coefficient matrix is not zero.
   - FACT 4: If determinant of the coefficient matrix of homogeneous system of linear equations is zero then there are infinitely many nontrivial solutions $(x, y) \neq (0, 0)$.

2. Use Facts 1-4 to determine if each the following systems of linear equations has one solution, no solutions, infinitely many solutions. Then find the solution/s (if any).

(a) $$\begin{aligned} 2x + 3y &= 5 \\ x - y &= 4 \end{aligned}$$

(b) $$\begin{aligned} 2x - 2y &= 4 \\ x - y &= 7 \end{aligned}$$

(c) $$\begin{aligned} 2x - 2y &= 0 \\ 3x + 3y &= 0 \end{aligned}$$

(d) $$\begin{aligned} 2x - 2y &= 0 \\ 3x - 3y &= 0 \end{aligned}$$