12. Linear homogeneous equations of second order with constant coefficients the case of distinct roots of characteristic equation (sec. 3.1)

1. PROBLEM: Find general solutions of linear homogeneous equation

$$ay'' + by' + cy = 0 \tag{1}$$

with constant real coefficients a, b, and c.

2. Recall that

   - By Superposition Principle: *Any* `linear combination` $C_1y_1(t) + C_2y_2(t)$ *of any two solutions* $y_1(t)$ *and* $y_2(t)$ *of (3) is itself a solution.*
   - The family of solutions $y(t) = C_1y_1(t) + C_2y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (3) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental set** of solutions of (3).

3. To find a fundamental set for the equation (3) note that the nature of the equation suggests that it may have solutions of the form

$$y = e^{rt}.$$

Plug in and get so called **characteristic equation** of (3):

$$ar^2 + br + c = 0. \tag{2}$$

Note that the characteristic equation can be determined from its differential equation simply by replacing $y^{(k)}$ with r^k.

4. *Solve* $y'' - 16y' = 0$.

5. `Fact from Algebra:` The quadratic equation $ar^2 + br + c = 0$ has roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which fall into one of 3 cases:

   - two distinct real roots $r_1 \neq r_2$ (in this case $D = b^2 - 4ac > 0$) [section 3.1]
   - two complex conjugate roots $r_1 = \overline{r_2}$ (in this case $D = b^2 - 4ac < 0$) [section 3.3]
   - two equal real roots $r_1 = r_2$ (in this case $D = b^2 - 4ac = 0$) [section 3.4]

Case 1: Two distinct real roots $(D = b^2 - 4ac > 0)$

6. Two distinct real roots r_1 and r_2 of the characteristic equation give us two solutions

$$y_1(t) = e^{r_1t}, \quad y_2(t) = e^{r_2t} \quad (r_1 \neq r_2).$$

Is this a fundamental set of solutions?

7. *Consider*

$$3y'' - y' - 2y = 0.$$

   (a) *Find general solution.*

   (b) *Find solution satisfying the following initial conditions:* $y(0) = \alpha, \quad y'(0) = 1$, *where* α *is a real parameter.*

   (c) *Find all* α *so that the solution of the corresponding IVP approaches* 0 *as* $t \to +\infty$.

8. *Consider*

$$y'' + (a+1)y' + (a-2)(1-2a)y = 0$$

where a is a real parameter.

(a) *Determine the values of the parameter a, if any, for which all solutions tend to zero as* $t \to \infty$.

(b) *Determine the value of the parameter a, if any, for which all (nonzero) solutions become unbounded as* $t \to \infty$.

13. The case of equal (or repeated) roots (section 3.4)

(a) **Case 3:** two equal/repeated real roots $r_1 = r_2$ (in this case $D = b^2 - 4ac = 0$)

(b) Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients

$$ay'' + by' + cy = 0 \tag{3}$$

is

$$ar^2 + br + c = 0. \tag{4}$$

$$D = b^2 - 4ac = 0 \Rightarrow r_1 = r_2 = -\frac{b}{2a}$$

So, we found one particular solution $y_1(t) = e^{r_1 t}$.

(c) How to find a second particular solution y_2 such that the set $\{y_1, y_2\}$ will be fundamental, i.e. $W(y_1, y_2) \neq 0$?

The method actually works both for distinct and repeated roots.

If r_1 and r_2 are roots of the characteristic equation (4), then

$$ar^2 + br + c = a(r - r_1)(r - r_2)$$

Then use "factorization" (in Leibnitz notation):

$$ay'' + by' + cy = a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = \left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b\frac{\mathrm{d}}{\mathrm{d}t} + c\right)[y] = a\left(\frac{\mathrm{d}}{\mathrm{d}t} - r_1\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} - r_2\right)[y]\right] = 0$$

(d) $\{y_1, y_2\} = \left\{e^{rt}, te^{rt}\right\}$ is fundamental set. Thus the general solution of (3) is

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

(e) Consider $y'' - 6y' + 9y = 0$.

i. Find general solution.

ii. Find solution subject to the initial conditions $y(0) = 2, y'(0) = \alpha$.

iii. For each α what is the behavior of the solutions as $t \to +\infty$?