16. Euler equations (beginning of sections 5.4 and problem 34 page 166 of the book)

1. In this section we discuss another class of linear homogeneous equations of second order for which the general solution can be found explicitly, the *Euler equations.* These are equation of the type

$$ax^2y'' + bxy' + cy = 0, \quad x > 0, \tag{1}$$

where a, b, c are real constants, $a \neq 0$

Look for a solution in the form $y(x) = x^r$ for some r. Substitute to the equation:

So we get the following equation, called the *indicial equation*:

$$\boxed{ar(r-1) + br + c = 0} \Leftrightarrow \boxed{ar^2 + (b-a)r + c = 0} \tag{2}$$

2. As in the case of equations with constant coefficients the most simple case is the case when the indicial equation has distinct real roots r_1 and r_2. In this case the set $\{x^{r_1}, x^{r_2}\}$ forms a fundamental set of solutions and the general solution is

$$y(x) = C_1x^{r_1} + C_2x^{r_2}.$$

EXAMPLE 1. *Find general solution of the equation* $x^2y'' + 2xy' - 6y = 0, \quad x > 0.$

3. In order to understand what is the form of general solution for other cases, i.e. when the roots of indicial equation are repeated or complex, we can reduce the equation (1) to the linear equation with constant coefficient by the following change of independent variable:

$$\boxed{x = e^t} \Leftrightarrow \boxed{t = \ln x} \tag{3}$$

Then by chain rule

$$\frac{dy}{dt} =$$

$$\frac{d^2y}{dt^2} =$$

So the equation (1) is transformed in the new independent variable t to the equation:

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0 \tag{4}$$

So, *the indicial equation for the Euler equation* (1) *coincides with the characteristic equation for the equation* (4) and **the general solution for the Euler equation** (1) **is obtained from the general solution to the equation** (4) **by replacing** t **with** $\ln x$ (subsequently e^{r_it} is replaced by x^{r_i}).

Using this rule, we can summarize all cases of the roots of the indicial equation in the following table (analogous to the table for the equations with constant coefficients):

SUMMARY:

Solution of the second order Euler equation

$$ax^2y'' + bxy' + cy = 0, \quad x > 0,$$

We consider the indicial equation

$$ar(r-1) + br + c = 0 \Leftrightarrow ar^2 + (b-a)r + c = 0$$

and consider its discriminant $D = (b-a)^2 - 4ac$.

Sign of $D = (b-a)^2 - 4ac$	Roots of indicial equation $ar^2 + br + c = 0$	General solution
$D > 0$	two distinct real roots $r_1 \neq r_2$	$y(x) = C_1x^{r_1} + C_2x^{r_2}$
$D < 0$	two complex conjugate roots $r_1 = \overline{r_2}$: $r_{1,2} = \lambda \pm i\mu$	$y(x) = x^\lambda\Big(C_1\cos(\mu\ln x) + C_2\sin(\mu\ln x)\Big)$
$D = 0$	two equal(repeated) real roots $r_1 = r_2 = r$	$y(x) = x^r(C_1 + C_2\ln x)$

17. Reduction of order (section 3.4 continued, very brief)

4. Consider second order linear homogeneous equation with arbitrary coefficients:

$$y'' + p(t)y' + q(t)y = 0 \tag{5}$$

Assume that we already know one of its particular solutions, $y_1(t)$. How to find another solution to get a fundamental set?

Step 1. Look for a second solution in the form

$$y(t) = v(t)y_1(t).$$

Step 2. Set $v' = w$ to reduce order and solve the obtained first order linear homogeneous ODE.

5. Note that the method can be also applied to linear nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t).$$

6. Consider the equation with constant coefficients

$$ay'' + by' + cy = 0$$

such that the discriminant $D = b^2 - 4ac$ is equal to zero. Assume that r is the (repeated) root of the characteristic equation. We now that $y_1(t) = e^{rt}$ is a solution of this equation. Find the second independent solution, using the method of reduction of order.

7.

REMARK 2. *Note that in the case of second order equation we also can use the Abel theorem to find the second independent solution of* (5) *from the knowledge of one solution* $y_1(t)$.