20. Basic Theory of Systems of First Order Linear Equations (sec. 7.4)

System of homogeneous linear equations

1. Consider a system of first order linear homogeneous DE:

$$X' = P(t)X. \tag{1}$$

Superposition Principle: *If the vector functions X_1 and X_2 are solutions of the homogeneous system (1), then the linear combination $C_1X_1 + C_2X_2$ is also a solution for any constants C_1, C_2.*

2. Given the following system

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

Show that the vector functions

$$X_1 = \begin{pmatrix} 2\cos t \\ -\cos t + \sin t \\ -2\cos t - 2\sin t \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

are solutions of the given system. Discuss a linear combination of these solutions.

3. Consider IVP

$$X' = P(t)X, \quad X(t_0) = b. \tag{2}$$

By Superposition Principle, if the vector functions

$$X_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \dots, X_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

are solutions of the homogeneous system (1), then the linear combination

$$X(t) = C_1 X_1(t) + \dots + C_n X_n(t)$$

is also a solution of (1) for any constants $C_1, \dots, C_n$.

Question: How to determine the the constants $C_1, \dots, C_n$ corresponding to the given IVP?

4. Consider the matrix, whose columns are vectors $X_1(t), \dots, X_n(t)$:

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix}.$$

Then $B = X(t_0) = C_1 X_1(t_0) + \dots + C_n X_n(t_0) = \Psi(t_0) \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$, equivalently,

$$\Psi(t_0) \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = b.$$

We can find a solution for any initial condition given by vector column b if and only if $det\Psi(t_0) \neq 0$.

EXPLANATION:

5. Note that this determinant is called the **Wronskian** of the solutions $X_1, \ldots, X_n$ and is denoted by
$$W\left[X_1, \ldots, X_n\right](t) = \det \Psi(t).$$

6. Note that by analogy with section 3.2, $det\Psi(t_0) \neq 0$ implies $det\Psi(t) \neq 0$ for any t.

7. If $det\Psi(t) \neq 0$ then $X_1, \ldots, X_n$ is called the **fundamental set of solutions** and the general solution of the system (1) is $C_1X_1(t) + \ldots + C_nX_n(t)$.

8. Given that the vector functions $X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ and $X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ are solutions of the system $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$. Find general solution of these system.

9. Question:*How to find a fundamental set of solutions?* In the next section we answer it for the case $P(t) = const$, i.e. for system of linear homogeneous equations with constant coefficients.

10. Find general solution of $X' = AX$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. (uncoupled system)

21. System of homogeneous linear equations with constant coefficients: why the eigenvalues and eigenvectors are important (sec 7.3 and 7.5)

1. A number λ is called an **eigenvalue** of matrix A if there exists a **nonzero** vector v such that
$$Av = \lambda v,$$
and v is called an **eigenvector** corresponding to the eigenvalue λ.

2. Example (corresponds to uncoupled systems). If A is a *diagonal matrix*,

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

are the corresponding eigenvectors.

3. **Fundamental Proposition.** *If λ is an eigenvalue of matrix A and v is an eigenvector corresponding to this eigenvalue then*

$$X(t) = e^{\lambda t} v$$

is a solution of the system $X' = AX$, i.e solution of the homogeneous linear system with constant coefficients.

Real Distinct Eigenvalues

4. FACT from Linear algebra: *If A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and $v^1, \dots, v^n$ are the corresponding eigenvectors, then* $\det(v^1, \dots, v^n)$ *(i.e. the determinant of the matrix with jth column equal to v^j does not vanish) or, equivalently the collection of vectors $v_1, \dots, v_n$ form a basis of $\mathbb{R}^n$*

As a consequence, if If A has distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\left\{e^{\lambda_1 t}v^1, \ldots, e^{\lambda_n t}v^n\right\}$$

is a fundamental set of solutions and the general solution is

$$X(t) = C_1 e^{\lambda_1 t}v^1 + \ldots + C_n e^{\lambda_n t}v^n. \tag{3}$$

REMARK If the eigenvalues are distinct but some of them are complex, the formula (3) gives the general complex-valued solutions, so we need to make additional work to get the general real-valued solutions similar to section 3.3 (will be discussed in section 7.6).

5. Eigenvalue are solutions of the following **characteristic equation (polynomial):**

$$\det(A - \lambda I) = 0.$$

6. Show that the characteristic equation in the case $n = 2$ can be found as

$$\lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = 0.$$

7. EXAMPLE. Consider the following system of DE:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned} \tag{4}$$

(a) Find general solution of (4).

(b) Find solution of (4) subject to the initial condition $X(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(c) What is behavior of the solution as $t \to +\infty$?