How to determine the shape of ellipses for the phase portrait of a center

Assume that the matrix A has complex eigenvalues $\pm i\beta$. Then the trajectories of the system $X' = AX$ are ellipses around the origin. Moreover, from linearity the ellipses are similar, i.e. they are obtained from one ellipse by homothety or , equivalently, they have the same axis of symmetries and the same ratio of semiaxis. The goal of this notes is to determine the shape of these ellipses, i.e. the axes of symmetries and the ratio of semiaxes.

For definiteness, assume that $\beta > 0$. Let $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ be an eigenvector of the eigenvalue $i\beta$. Form the following 2×2 symmetric matrix Γ, called the *Gram matrix* of vectors $\mathbf{a}, \mathbf{b}$:

$$\Gamma = \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{pmatrix}, \tag{1}$$

where $\cdot$ denotes the dot product. Analogously one can define the Gram matrix for any n vectors, as an $n \times n$ matrix consisting of all possible dot products of these vectors.

As we briefly discussed a symmetric matrix have a basis of eigenvectors, moreover, the eigenspaces, corresponding to different eigenvalues, are orthogonal. Besides, all eigenvalues of symmetric matrices are real. Moreover, the Gram matrix of linearly independent vectors has positive eigenvalues.

PROPOSITION 1 (Formulation of the rules to determine the shape of the trajectories of $X' = AX$). *The following statements hold:*

1. *If* $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ *is an eigenvector of* Γ *then the line generated by the vectir* $\alpha_1\mathbf{a} + \alpha_2\mathbf{b}$ *is the axis of symmetries of the ellipses;*
2. *the ratio of semiaxes of the ellipses is equal to the square root of the ration of the corresponding eigenvalues. In particular, if* Γ *has repeated eigenvalues then the ellipses are circles.*

In other words, to find the direction of axes of symmetry you have first to find the eigenvectors of the Gram matrix Γ and then to use them as coefficients in linear combination of $\mathbf{a}$ and $\mathbf{b}$ and to find the ration of semiaxes you have to find the eigenvalues of Γ and to take the square root of their ratio.

PROOF:

1. As was already mentioned the pair of vectors $(\mathbf{a}, \mathbf{b})$ is not uniquely defined: we can multiply the eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ of $i\beta$ by any complex number, say $re^{i\phi}$ to get another eigenvector $\tilde{\mathbf{v}} = \tilde{\mathbf{a}} + i\tilde{\mathbf{b}}$ of $i\beta$ with another pair of vectors $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ which satisfy

$$\begin{aligned} \widetilde{\mathbf{a}} &= r(\cos\phi\,\mathbf{a} - \sin\phi\,\mathbf{b}) \\ \widetilde{\mathbf{b}} &= r(\sin\phi\,\mathbf{a} + \cos\phi\,\mathbf{b}) \end{aligned} \tag{2}$$

2. Let $\widetilde{\Gamma}$ be the Gram matrix of the pair of vectors $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$. Find he relation between the matrices $\widetilde{\Gamma}$ and Γ. For this let

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

The transformation $\mathbf{x} \mapsto R_\phi \mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$ corresponds to the rotation of the plane in angle $-\phi$.

Now if Φ is 2×2 matrix with the first column equal to $\mathbf{a}$ and the second columns equal to $\mathbf{b}$ and similarly $\widetilde{\Phi}$ is 2×2 matrix with the first column equal to $\tilde{\mathbf{a}}$ and the second columns equal to $\tilde{\mathbf{b}}$, then (2) can be rewritten as

$$\widetilde{\Phi} = r\Phi R_\phi \tag{3}$$

Besides $\Gamma = \Phi^T\Phi$ and $\widetilde{\Gamma} = \widetilde{\Phi}^T\widetilde{\Phi}$ (here B^T denotes the transpose of a matrix B).Combining this with (3) we get

$$\widetilde{\Gamma} = r^2(R_\phi)^T\Gamma R_\phi = r^2 R_\phi^{-1}\Gamma R_\phi \tag{4}$$

(here we use that $R_\phi^T = R_\phi^{-1}$). As you will learn in Linear Algebra, this means that matrices Γ and $\widetilde{\Gamma}$ are similar and they represent the same linear transformation. In particular they have the same eigenvalues.

3. Now we refer again to an important general theorem in Linear Algebra about orthogonal diagonalization of symmetric matrices (although in the considered case $n = 2$ and it can be obtained by elementary calculations). According to this theorem applied to our case we can find ϕ such that $\widetilde{\Gamma}$ will be diagonal. This means that we can find ϕ such that the corresponding pair of vector $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are orthogonal, because the diagonality of $\widetilde{\Gamma}$ is equivalent to $\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} = 0$. So,

$$\Gamma = \begin{pmatrix} \tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}} & 0 \\ 0 & \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} |\tilde{\mathbf{a}}|^2 & 0 \\ 0 & |\mathbf{b}|^2 \end{pmatrix}. \tag{5}$$

If we make our analysis as in section 27 in the basis $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$, then the trajectories are ellipses with axis of symmetries being the lines generated by vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ and the ratio of semiaxes equal to the ratio of magnitudes of these vectors. On the other hand, by (5) the squares of magnitudes of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are the eigenvalues of the matrix $\widetilde{\Gamma}$ and therefore of Γ itself, which proves the second statement of the proposition. To prove the first statement note that (4) implies that

$$\Gamma = \frac{1}{r^2} R_\phi \widetilde{\Gamma} R_\phi^{-1},$$

which in turn yields that the columns of the matrix R_ϕ are the eigenvectors of Γ. Then , using right hand-sides of (2) and the fact that $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ generates the axes of symmetries of the ellipses, we get the first statement of proposition. $\square$.