Examples of equations that are initially not separable but can be reduced to separable.

1. **Type 1** Homogeneous (nonlinear) equation (do not confuse with linear homogeneous equation discussed in the first week and that will be discussed later) are equations of the type

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \tag{1}$$

where F is a given function of a single variable. In other words, the right-hand side of the equation depends on the ratio $\frac{y}{x}$ only.

Make a substitution

$$u(x) = \frac{y(x)}{x}$$

.

Then $y(x) = xu(x)$. Substitute into equation (1):

The left-hand side is $y' = xu'(x) + u$ The right-hand side is $F(y/x) = F(u)$

Then substituting this into the original equation (1) we get

$$xu'(x) + u = F(u) \quad \Leftrightarrow \quad xu'(x) = F(u) - u$$

which is separable. Indeed,

$$\frac{du}{F(u) - u} = \frac{dx}{x}$$

EXAMPLE 1. *Reduce the equation* $\frac{dy}{dx} = \frac{3x + 2y}{2x - 3y}$ *to a separable equation*

Solution

Let $u(x) = \frac{y(x)}{x}$. Dividing the numerator and denominator of $\frac{3x + 2y}{2x - 3y}$ by x we get

$$\frac{3x + 2y}{2x - 3y} = \frac{3 + 2\frac{y}{x}}{2 - 3\frac{y}{x}} = \frac{3 + 2u}{2 - 3u}$$

Then substituting u into the original equation (1) we get

$$xu'(x) + u = \frac{3 + 2u}{2 - 3u}$$

which implies

that

$$xu'(x) = \frac{3 + 2u}{2 - 3u} - u = \frac{3 + \cancel{2u} - \cancel{2u} + 3u^2}{2 - 3u}$$

so we separate the variables in the equation as follows:

$$\frac{(2-3u)du}{3u^2+3} = \frac{dx}{x}$$

Then we can integrate it to find the general solution, using the method of integration of rational function studies in Calculus 2.

2. **Type 2** Equations of the type

$$\frac{dy}{dx} = F(ax+by+c), \tag{2}$$

where F is a given function of a single variable and a, b, c are constants, $b \neq 0$.

Make a substitution

$$u(x) = ax + by(x) + c$$

.

Then $y(x) = \frac{1}{b}(u(x) - ax - c)$. Substitute into equation (2):

The left-hand side is

$$y' = \frac{1}{b}(u'(x) - a)$$

. The right-hand side is $F(ax + by + c) = F(u)$

Then plug it into (2) we ge

$$\frac{1}{b}(u' - a) = F(u) \quad \Leftrightarrow \quad u' = bF(u) + a$$

which is a separable equation.

EXAMPLE 2. *Reduce the equation* $\frac{dy}{dx} = \frac{1}{2y+x}$ *to a separable equation*

Solution

Let $u(x) = 2y(x) + x$. Then $y(x) = \frac{1}{2}\big(u(x) - x\big)$ and $y'(x) = \frac{1}{2}\big(u'(x) - 1\big)$. So substituting to the original equation we get

$$\frac{1}{2}\big(u'(x) - 1\big) = \frac{1}{u}$$

or, equivalently,

$$u'(x) = \frac{2}{u} + 1 = \frac{2+u}{u}$$

So,

$$\frac{udu}{2+u} = dx.$$

Then we can integrate it to find the general solution.