Nonexact equation that can be made exact using integrating factors depending on single variable (section 2.6 continued)

1. Consider again a first order differential equation

$$P(x,y) + Q(x,y)y' = 0 \quad (\text{or} \quad P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = 0) \tag{1}$$

on a simply connected region R of $\mathbb{R}^2$. and assume that it is not exact, i.e. $P_y \neq Q_x$ on R.

Goal To find a function μ such that after multiplication of (1) by μ the equation becomes exact. Such function μ is called an *integrating factor for equation* (1). For this we need that

$$(\mu P)_y = (\mu Q)_x$$

Using the product rule we get:

$$P\mu_y + P_y\mu = Q\mu_x + Q_x\mu \Leftrightarrow$$

$$P\mu_y - Q\mu_x + (P_y - Q_x)\mu = 0. \tag{2}$$

This is a first order linear *partial* differential equation (PDE) for the function μ and to solve it is equally hard as to solve the original equation (1). So, in general, the idea of making equation (1) exact does not give an efficient method to solve it.

However, in some specific cases, this idea works perfectly.

(a) For example, suppose we can find the integrating factor μ which is a function of x alone

QUESTION: Under what condition on P and Q is it possible?

ANSWER: Integrating factor $\mu = \mu(x)$ (i.e depending on x alone) exists if and only if $\dfrac{P_y - Q_x}{Q}$ is a function of x alone. In this case this integrating factor satisfies the ordinary differential equation

$$\frac{d\mu}{dx} = \frac{P_y - Q_x}{Q}\mu. \tag{3}$$

Note that this is a linear homogeneous equation, in particular, it is separable and we know how to solve it by the method of section 2.2

EXPLANATION:

If μ is a function of x alone then $\mu_y = 0$. Plugging this into (2) we get

$$-Q\mu_x + (P_y - Q_x)\mu = 0 \Leftrightarrow$$

$$\mu_x = \frac{P_y - Q_x}{Q}\mu. \tag{4}$$

Since μ depends on x only, then from the last equation the expression $\frac{P_y - Q_x}{Q}$ must depend on x. Note also that by the same reason we can write μ_x in (4) as $\frac{d\mu}{dx}$ to get (3).

Vice versa, by reversing the arguments above , if $\frac{P_y - Q_x}{Q}$ depends on x only, then any function μ satisfying the differential equation (3) will serve as an integrating factor such that after multiplying (1) by such μ we get an exact equation.

(b) Similarly, integrating factor $\mu = \mu(y)$ (i.e depending on y alone) exists if and only if $\frac{P_y - Q_x}{P}$ is a function of y alone. In this case this integrating factor satisfies the equation

$$\frac{d\mu}{dy} = \frac{Q_x - P_y}{P}\mu. \tag{5}$$

EXPLANATION:

If μ is a function of y alone then $\mu_x = 0$. Plugging this into (2) we get

$$P\mu_x + (P_y - Q_x)\mu = 0 \Leftrightarrow$$

$$\mu_y = \frac{Q_x - P_y}{P}\mu. \tag{6}$$

Since μ depends on x only, then from the last equation the expression $\frac{Q_x - P_y}{P}$ must depend on y. Note also that by the same reason we can write μ_y in (6) as $\frac{d\mu}{dy}$ to get (5).

Vice versa, by reversing the arguments above , if $\frac{Q_x - P_y}{P}$ depends on y only, then any function μ satisfying the differential equation (5) will serve as an integrating factor such that after multiplying (1) by such μ we get an exact equation.

2. **EXAMPLE** Solve

$$(3x^2y + 2xy + y^3)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0 \tag{7}$$

SOLUTION:

Note that this is the same equation as in section 5 , problem 4b and we already checked that it is not exact. In this case

$$\begin{aligned} P &= 3x^2y + 2xy + y^3 &\Rightarrow\quad & P_y = 3x^2 + 2x + 3y^2; \\ Q &= x^2 + y^2 &\Rightarrow\quad & Q_x = 2x \end{aligned}$$

Verify whether the expression $\dfrac{P_y - Q_x}{Q}$ depends on x only (or equivalently is independent of y):

$$\frac{P_y - Q_x}{Q} = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2 + y^2} = 3$$

so it is constant and in particularly it is independent of y. Hence we can look for an integrating factor $\mu = \mu(x)$ and it satisfies the differential equation (3) which in this case is

$$\mu'(x) = 3\mu(x)$$

(recall that the coefficient 3 here comes from our calculation that $\dfrac{P_y - Q_x}{Q} = 3$). So, we can take μ as

$$\mu(x) = e^{3x}. \tag{8}$$

REMARK 1. *If it turned out that* $\dfrac{P_y - Q_x}{Q}$ *depends on* y*, then you have to check that* $\dfrac{Q_x - P_y}{P}$ *is independent of* x*. If yes, then you look for an integrating factor depending on* y *solving* (5)*. If not, then the method described above does not work.*

Now multiply both sides of equation (7) by $\mu(x) = e^{3x}$ to get an exact equation

$$e^{3x}(3x^2y + 2xy + y^3)\mathrm{d}x + e^{3x}(x^2 + y^2)\mathrm{d}y = 0$$

For this new equation $P = e^{3x}(3x^2y + 2xy + y^3)$ and $Q = e^{3x}(x^2 + y^2)$.

Now find a potential Φ by solving the system

$$\Phi_x = e^{3x}(3x^2y + 2xy + y^3) \tag{9}$$

$$\Phi_y = e^{3x}(x^2 + y^2) \tag{10}$$

Note that it is much more convenient to integrate first (10) rather than (9) (the latter , for example, will need integration by parts). Integrating (10) with respect to y we get:

$$\Phi = \int (e^{3x}x^2 + e^{3x}y^2)\,dx = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + g(x)$$

for some function $g(x)$. Then plug this Φ into equation (9):

$$\frac{\partial}{\partial x}\big(e^{3x}(x^2y + \frac{y^3}{3}) + g(x)\big) = e^{3x}(3x^2y + 2xy + y^3) \tag{11}$$

The left hand-side of (11) is equal to

$$3e^{3x}(x^2y + \frac{y^3}{3}) + e^{3x}(2xy) + g'(x) = e^{3x}(3x^2y + 2xy + y^3) + g'(x).$$

Equating this with the right hand-side of (11) we get $g'(x) = 0$. So we can take $g(x) = 0$. So a potential can be taken as

$$\Phi(x, y) = e^{3x}\left(x^2y + \frac{y^3}{3}\right)$$

Therefore the general solution of (7) is given implicitly by

$$\boxed{e^{3x}\left(x^2y + \frac{y^3}{3}\right) \equiv C}.$$

In the next two items it is shown that in fact all previous discussed types of equations can be solved by the method of item 1:

3. **Relation to separable equations.**

Separable equation have the form

$$\frac{dy}{dx} = f(y)g(x) \Leftrightarrow f(y)g(x)dx - dy = 0 \tag{12}$$

Hence $P = f(y)g(x), Q = 1$. So, $P_y = f'(y)g(x), Q_x = 0$, and the expression

$$\frac{Q_x - P_y}{Q} = -\frac{f'(y)\cancel{g(x)}}{f(y)\cancel{g(x)}} = -\frac{f'(y)}{f(y)}$$

depends on y only. So we can look for an integrating factor $\mu = \mu(y)$ satisfying $\mu' = -\dfrac{f'(y)}{f(y)}\mu$ which implies we can take $\mu(y) = \frac{1}{f(y)}$. Multiplying (12) by $\dfrac{1}{f(y)}$ we get an exact equation

$$g(x)dx - \frac{1}{f(y)}dy = 0$$

This can be obtained without any theory, I just wanted to demonstrate the method once more on this example.

4. **Relation to the method of integrating factor for linear equations from section 2.1** Let us show that the method of integrating factor for linear equations from section 2.1 is a particular case of the method discussed here.

   In section 2.1 of the textbook (or section 3 of our notes) we discussed linear equations of type

$$y' + p(x)y = g(x) \Leftrightarrow \big(p(x)y - g(x)\big)dx + dy$$

   Hence,

$$\begin{aligned} &P(x, y) = p(x)y - g(x) \;\Rightarrow\; P_y = p(x) \\ &Q(x, y) = 1 \;\Rightarrow\; Q_x = 0 \end{aligned}$$

   Then the expression

$$\frac{P_y - Q_x}{Q} = p(x)$$

   depends on x only, so one can look for an integrating factor $\mu = \mu(x)$ and te differential equation (3) takes the form

$$\mu' = p(x)\mu,$$

   which is exactly as in section 2.1. It shows that the method of integrating factor of section 2.1 is in fact a very particular case of the method of integrating factor presented in item 1 of this section.