Abel’s theorem for Wronskian of solutions of linear homogeneous systems and higher order equations

Recall that the trace $\text{tr}(A)$ of a square matrix $A$ is the sum its diagonal elements.

**THEOREM 1. (Abel’s theorem for first order linear homogeneous systems of differential equations)** Assume that $n$ vector functions $X_1(t), \ldots, X_n(t)$ are solutions of a first order linear homogeneous system of $n$ ODEs

$$X' = P(t)X.$$  

and $W(t) := W(X_1, \ldots, X_n)(t)$ is the Wronskian of the solutions $X_1, \ldots, X_n(t)$. Then

$$W'(t) = \text{tr}(P(t))W(t).$$

**Proof for $n = 2$** For simplicity prove the theorem in the case $n = 2$ only (the proof for arbitrary $n$ is exactly the same, just one works with $n \times n$ matrices instead $2 \times 2$). Assume that

$$X_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}, \quad X_1(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

Then

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Note that if an $n \times n$ matrix $A(t)$ depends on a variable $t$, then, as a consequence of a product rule for derivatives, the derivative of its determinant with respect to $t$ is equal to the sum of $n$ terms such that the $i$th term is equal to the determinant of the matrix obtained from $A(t)$ by replacing the $i$th row by the derivative of the $i$th row of $A$. Therefore

$$W'(t) = \begin{vmatrix} x'_{11}(t) & x'_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x'_{21}(t) & x'_{22}(t) \end{vmatrix}.$$  

(3)

Since $X_1(t)$ and $X_2(t)$ are solutions of (1), the first determinant in (3) can be represented as

$$\begin{vmatrix} x'_{11}(t) & x'_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = \begin{vmatrix} p_{11}(t)x_{11}(t) + p_{12}(t)x_{21}(t) & p_{11}(t)x_{12}(t) + p_{12}(t)x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} =$$

$$= \begin{vmatrix} p_{11}(t)x_{11}(t) & p_{11}(t)x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} p_{12}(t)x_{21}(t) & p_{12}(t)x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} =$$

$$= p_{11}(t)\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + p_{12}(t)\begin{vmatrix} x_{21}(t) & x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = p_{11}(t)W(t)$$  

(4)

(the second determinant in (4) is equal to zero because the rows of the matrix are identical).
In completely similar way the second determinant of (3) can be represented as
\[
\begin{vmatrix}
  x_{11}(t) & x_{12}(t) \\
  x_{21}'(t) & x_{22}'(t)
\end{vmatrix} = p_{22}(t)W(t)
\]  
(5)

Plugging (4) and (5) into (3) we get
\[
W'(t) = (p_{11}(t) + p_{22}(t))W(t) = \text{tr}(P(t))W(t),
\]
which proves our theorem in the case \(n = 2\)

**COROLLARY 2. (Abel’s theorem for second order linear homogeneous equations)** If \(y_1(t)\) and \(y_2(t)\) are solutions of linear homogeneous ODE of second order
\[
y'' + p(t)y' + q(t)y = 0
\]  
(6)

and \(W(t) := W(y_1, y_2)(t)\) is the Wronskian of \(y_1(t)\) and \(y_2(t)\), then
\[
W'(t) + p(t)W(t) = 0
\]  
(7)

**Proof.** Indeed the matrix \(P(t)\) of the system of first order equations corresponding to the second order equation (8) is
\[
P(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}.
\]
So,
\[
\text{tr}(P(t)) = -p(t).
\]
Hence by (2), \(W'(t) = -p(t)W(t)\), which is equivalent to (7).

Here is the direct generalization of Corollary 2 to \(n\)th order linear homogeneous equations

**COROLLARY 3. (Abel’s theorem for \(n\)th order linear homogeneous ODE)** If \(y_1(t), \ldots, y_n(t)\) are solutions of a linear homogeneous ODE of order \(n\)
\[
y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y(t) + p_0(t)y(t) = 0
\]  
(8)

and \(W(t) := W(y_1, \ldots, y_n)(t)\) is the Wronskian of \(y_1(t), \ldots, y_n(t)\), then
\[
W'(t) + p_{n-1}(t)W(t) = 0
\]

Try to deduce it from Theorem 1 by looking for the trace of the matrix of the corresponding system.