Abel's theorem for Wronskian of solutions of linear homogeneous systems and higher order equations

Recall that the trace $\operatorname{tr}(A)$ of a square matrix A is the sum its diagonal elements.

THEOREM 1. *(Abel's theorem for first order linear homogeneous systems of differential equations) Assume that n vector functions $\mathbf{X}_1(t), \dots, \mathbf{X}_n(t)$ are solutions of a first order linear homogeneous system of n ODEs*

$$\mathbf{X}' = P(t)\mathbf{X}. \tag{1}$$

and $W(t) := W(\mathbf{X}_1, \dots, \mathbf{X}_n)(t)$ is the Wronskian of the solutions $\mathbf{X}_1, \dots, \mathbf{X}_n(t)$. Then

$$W'(t) = \operatorname{tr}\big(P(t)\big)W(t). \tag{2}$$

Proof for $n = 2$ For simplicity prove the theorem in the case $n = 2$ only (the proof for arbitrary n is exactly the same, just one works with $n \times n$ matrices instead 2×2). Assume that

$$\mathbf{X}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}, \mathbf{X}_1(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

Then

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Note that if an $n \times n$ matrix $A(t)$ depends on a variable t, then, as a consequence of a product rule for derivatives, the derivative of its determinant with respect to t is equal to the sum of n terms such that the ith term is equal to the determinant of the matrix obtained from $A(t)$ by replacing the ith row by the derivative of the ith row of A. Therefore

$$W'(t) = \begin{vmatrix} x'_{11}(t) & x'_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x'_{21}(t) & x'_{22}(t) \end{vmatrix}. \tag{3}$$

Since $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are solutions of (1), the first determinant in (3) can be represented as

$$\begin{vmatrix} x'_{11}(t) & x'_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = \begin{vmatrix} p_{11}(t)x_{11}(t) + p_{12}(t)x_{21}(t) & p_{11}(t)x_{12}(t) + p_{12}(t)x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} =$$

$$\begin{vmatrix} p_{11}(t)x_{11}(t) & p_{11}(t)x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} p_{12}(t)x_{21}(t) & p_{12}(t)x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} =$$

$$p_{11}(t) \underbrace{\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}}_{W(t)} + p_{12}(t) \underbrace{\begin{vmatrix} x_{21}(t) & x_{22}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}}_{0} = p_{11}(t)W(t) \tag{4}$$

(the second determinant in (4) is equal to zero because the rows of the matrix are identical).

In completely similar way the second determinant of (3) can be represented as

$$\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x'_{21}(t) & x'_{22}(t) \end{vmatrix} = p_{22}(t)W(t) \tag{5}$$

Plugging (4) and (5) into (3) we get

$$W'(t) = \big(p_{11}(t) + p_{22}(t)\big)W(t) = \operatorname{tr}\big(P(t)\big)W(t),$$

which proves our theorem in the case $n = 2$

COROLLARY 2. *(Abel's theorem for second order linear homogeneous equations) If $y_1(t)$ and $y_2(t)$ are solutions of linear homogeneous ODE of second order*

$$y'' + p(t)y' + q(t)y = 0 \tag{6}$$

and $W(t) := W(y_1, y_2)(t)$ is the Wronskian of $y_1(t)$ and $y_2(t)$, then

$$W'(t) + p(t)W(t) = 0 \tag{7}$$

Proof. Indeed the matrix $P(t)$ of the system of first order equations corresponding to the second order equation (8) is

$$P(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}.$$

So,

$$\operatorname{tr}\big(P(t)\big) = -p(t).$$

Hence by (2), $W'(t) = -p(t)W(t)$, which is equivalent to (7). □

Here is the direct generalization of Corollary 2 to nth order linear homogeneous equations

COROLLARY 3. *(Abel's theorem for nth order linear homogeneous ODE) If $y_1(t), \dots, y_n(t)$ are solutions of a linear homogeneous ODE of order n*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y(t) + p_0(t)y(t) = 0 \tag{8}$$

and $W(t) := W(y_1, \dots, y_n)(t)$ is the Wronskian of $y_1(t), ldots, y_n(t)$, then

$$W'(t) + p_{n-1}(t)W(t) = 0$$

Try to deduce it from Theorem 1 by looking for the trace of the matrix of the corresponding system.