Extra credit problems on volumes of n-dimensional balls, MATH251

The n-dimensional ball of radius R with center at the origin is the set

$$\{(x_1,\dots,x_n)|x_1^2+x_2^2+\dots+x_n^2\leq R^2\}.$$

It is a straightforward generalization of the disk of radius R in the plane (corresponding to $n=2$) and of the usual ball of radius R in the 3-dimensional space. Note that for $n=1$ the ball of radius R around the origin is the segment $[-R,R]$ in $\mathbb{R}$.

The definition of the volume of an n-dimensional solid is a straightforward generalization of the definition of the area of a region in $\mathbb{R}^2$ via the double integral of 1 over it and of a volume of 3-dimensional solid via the triple integral of 1 over it (think how to define multi-dimensional integrals by analogy with single, double, and triple integrals).

Let $V_n(R)$ be the volume of an n-dimensional ball of radius R. For example, $V_1(R)=2R$ (the length of the segment $[-R,R]$ is equal to $2R$), $V_2(R)=\pi R^2$ (the area of the disk of radius R is equal πR^2), $V_3(R)=\frac{4}{3}\pi R^3$ (the volume of the usual ball of radius R in $\mathbb{R}^3$ is equal to $\frac{4}{3}\pi R^3$).

Below is a sequence of problems about $V_n(R)$. Attempt as many of them as you can.

a) Prove that
$$V_n(R)=\int_{-R}^{R}V_{n-1}\left(\sqrt{R^2-x^2}\right)dx,\quad n>1.$$

b) Prove that
$$V_n(R)=\int\int_{x_1^2+x_2^2\leq R^2}V_{n-2}\left(\sqrt{R^2-x_1^2-x_2^2}\right)dx_1\,dx_2,\quad n>2.$$

c) Prove that
$$V_n(R)=2\pi\int_0^R V_{n-2}\left(\sqrt{R^2-r^2}\right)rdr,\quad n>2.$$

Hint: use the change to polar coordinates in (x_1,x_2)-plane.

d) Prove that
$$V_n(R)=\pi\int_0^{R^2}V_{n-2}(\sqrt{u})du,\quad n>2.$$

e) Using item d) and the fact that $V_1(R)=2R$ prove that $V_3(R)=\frac{4}{3}\pi R^3$.

f) Using items c) or d) and the fact that $V_2(R)=\pi R^2$ prove that
$$V_4(R)=\frac{\pi^2}{2}R^4,$$

i.e. the volume of the 4-dimensional ball of radius R is equal to $\frac{\pi^2}{2}R^4$.

g) Using item d) and the fact that $V_3(R) = \frac{4}{3}\pi R^3$ prove that

$$V_5(R) = \frac{8\pi^2}{15}R^5,$$

i.e. the volume of the 5-dimensional ball of radius R is equal to $\frac{8\pi^2}{15}R^5$.

h) In general $V_n(R) = C_n R^n$, where C_n is the volume of n-dimension ball of radius 1 (justify!). For example $C_1 = 2$, $C_2 = \pi$, $C_3 = \frac{4}{3}\pi$, $C_4 = \frac{\pi^2}{2}$ (see item f)), $C_5 = \frac{8\pi^2}{15}$ (see item g)). Using item d) prove that

$$C_n = \frac{2\pi}{n}C_{n-2}.$$

i) Using the fact that $C_1 = 2$ and item h) appropriate number of times find C_7, C_9, and C_{11} (the volume of unit balls of dimension 7, 9, and 11)

j) Using the fact that $C_2 = \pi$ and item h) appropriate number of times find C_6, C_8, and C_{10} (the volume of unit balls of dimension 6, 8, and 10)

k) Using item h) and the fact that $C_1 = 2$ and $C_2 = \pi$ prove by induction that

$$C_{2l} = \frac{\pi^l}{l!}, \quad l \geq 1,$$
$$C_{2l+1} = \frac{2^{l+1}\pi^l}{(2l+1)!!}, \quad l \geq 0.$$

Here $l!$ denotes the product of all integers from 1 to l and $(2l+1)!!$ denotes the product of all odd integers from 1 to $2l+1$.