F19_LN_12_1

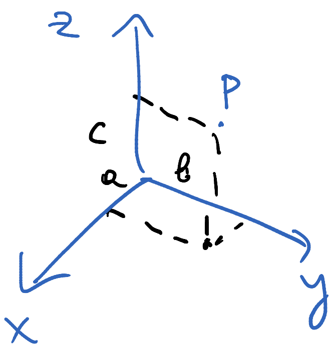
Monday, August 26, 2019 1:06 AM

12 Vectors and Geometry of Space

12.1: Three-dimensional Coordinate System

The three-dimensional coordinate system consists of the **origin** O and the **coordinate axes**: x-axis, y-axis, z-axis.The coordinate axes determine 3 **coordinate planes**: the xy-plane, the xz-plane and yz-plane. The coordinate planes divide space into 8 parts, called octants.

Representation of point $P(a,b,c)$ and its projections on the coordinate planes:



EXAMPLE 1. *Describe in words the regions of* $\mathbb{R}^3$ *represented by the following equation:*

(a) $z=0$ The set of all points $(x,y,0)$, i.e., xy-plane

(b) $y=0$ The set of all point $(x,0,z)$, i.e., xz-plane

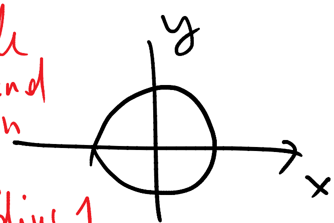
(c) $x=0$ yz-plane

Note that in $\mathbb{R}^2$ the graph of the equation involving x and y is a curve. In $\mathbb{R}^3$ an equation in x,y,z represents a **surface**.(It does not mean that we can't graph curves in $\mathbb{R}^3$.)

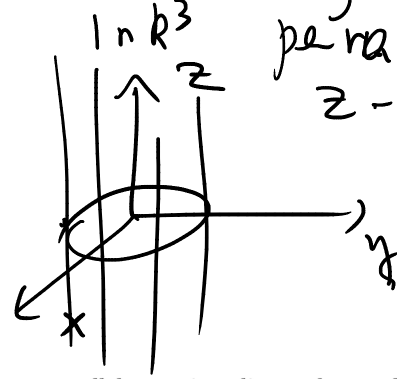
EXAMPLE 2. *Sketch the graph of $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.*

In $\mathbb{R}^2$

Circle around origin with radius 1



In $\mathbb{R}^3$



A cylinder parallel to z-axis with circular base

$x^2+y^2-1=0 \iff x^2+y^2=1$

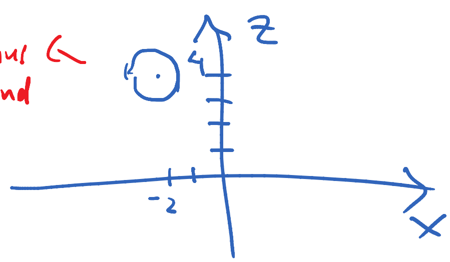
A **cylinder** is a surface of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

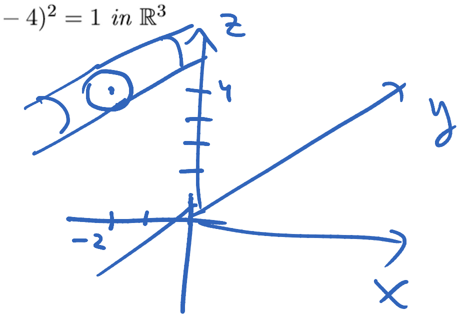
An equation that contains only two of the variables x, y, z represents a **cylindrical surface** *in $\mathbb{R}^3$. How to graph cylindrical surface:*

1. *graph the equation in the coordinate plane of the two variables that appear in the given equation;*
2. *translate that graph parallel to the axis of the missing variable.*

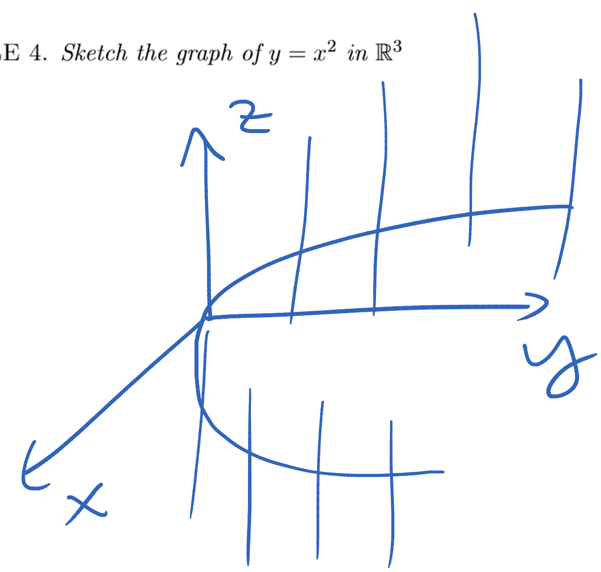
EXAMPLE 3. *Sketch the graph of $(x + 2)^2 + (\cancel{y}\, z - 4)^2 = 1$ in $\mathbb{R}^3$*

Circle of radius 1 around (-2,4)





EXAMPLE 4. *Sketch the graph of $y = x^2$ in $\mathbb{R}^3$*



EXAMPLE 5. *Let S be the graph of $x^2 + z^2 - 10z + 21 = 0$ in $\mathbb{R}^3$.*

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

(a) *Describe S.*

$x^2 + z^2 - 10z + 21 = 0$

Completing squares: $x^2 + \underbrace{(z^2 - 2\cdot 5\, z + 25)}_{(z-5)^2} - 25 + 21 = 0 \iff$

$x^2 + (z-5)^2 - 4 = 0 \iff x^2 + (z-5)^2 = 4 \to$ S is a cylinder parallel to y-axis with the base which is a circle of radius 2 around $(0,0,5)$ in xz-plane

(b) *The intersection of S with the xz- plane is* the same circle

$\begin{cases} y = 0 \\ x^2 + (z-5)^2 = 4 \end{cases}$

(c) *The intersection of S with the yz- plane is* two lines $(0, y, 7)$ & $(0, y, 3)$

$\begin{cases} x = 0 \\ x^2 + (z-5)^2 = 4 \end{cases} \Rightarrow (z-5)^2 = 4 \iff$ Either $z - 5 = 2 \Rightarrow z = 7$ or $z - 5 = -2 \Rightarrow z = 3$

(d) *The intersection of S with the xy- plane is* empty set

$\begin{cases} z = 0 \\ x^2 + (z-5)^2 = 4 \end{cases} \Rightarrow x^2 + 25 = 4 \Rightarrow x^2 = -21 \to$ no solutions.

Spheres

- **Distance formula in $\mathbb{R}^3$**: The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Based on Pythagorean Thm used twice

EXAMPLE 6. *Show that the equation $x^2 + y^2 + z^2 + 2x - 4y + 8z + 17 = 0$ represents a sphere, and find its center and radius.*

$\underbrace{(x^2 + \underset{2\cdot 1}{2x} + 1)}_{(x+1)^2} - 1 + \underbrace{(y^2 - \underset{2\cdot 2}{4y} + 4)}_{(y-2)^2} - 4 + \underbrace{(z^2 + \underset{2\cdot 4}{8z} + \underset{4^2}{16})}_{(z+4)^2} - 16 + 17 = 0$

$(x+1)^2 + (y-2)^2 + (z+4)^2 - 4 = 0 \iff (x+1)^2 + (y-2)^2 + (z+4)^2 = 4$

A sphere of radius 2 around $(-1, 2, -4)$.

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k$$

- If $k > 0$ then the graph of this equation is a sphere of radius $\sqrt{k}$ around (a,b,c)
- If $k = 0$, then the graph is a point (a,b,c)
- If $k < 0$ then empty set

Regions in $\mathbb{R}^3$

EXAMPLE 7. *Describe the set of all points in $\mathbb{R}^3$ whose coordinates satisfy the following inequality:* $x^2 + y^2 < 16$

$x^2+y^2=16$ is a cylinder parallel to z-axis with the base which is circle of radius 4 around the origin in xy-plane $\Rightarrow$
$x^2+y^2<16$ is the interior of this cylinder (not including the boundary)

EXAMPLE 8. *Describe the following region:* $\{(x,y,z)|9 \le x^2+y^2+z^2 \le 16\}$

The region between two spheres of radiuses 3 and 4 with centers at the origin (including the boundary spheres).