

12.2 & 12.3: Vectors and the Dot Product

DEFINITION 1. A **3-dimensional vector** is an ordered triple $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector \mathbf{a} with representation \overrightarrow{PQ} is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P .

EXAMPLE 2. Find the vector represented by the directed line segment with the initial point $A(1, 2, 3)$ and terminal point $B(3, 2, -1)$. What is the position vector of the point A ?

Vector Arithmetic: Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha \mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW

PARALLELOGRAM LAW

Two vectors \mathbf{a} and \mathbf{b} are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha \mathbf{a}$. Equivalently:

$$\mathbf{a} \parallel \mathbf{b} \Leftrightarrow$$

The magnitude or length of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|\mathbf{0}| = 0$.

Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

Unit vector in the same direction as \mathbf{a} : $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vector \mathbf{a} by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing \mathbf{a}** .*

Note that in \mathbb{R}^2 a nonzero vector \mathbf{a} can be determined by its length and the angle from the positive x -axis:

In \mathbb{R}^2 and \mathbb{R}^3 a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}| \hat{\mathbf{a}},$$

i.e. \mathbf{a} is equal to its length times a unit vector in the same direction.

EXAMPLE 3. Find the components of a vector \mathbf{a} of length $\sqrt{5}$ that extends along the line through the points $M(2, 5, 0)$ and $N(0, 0, 4)$.

Standard Basis Vectors:

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle =$$

EXAMPLE 4. Given $\mathbf{a} = \langle 1, 0, -3 \rangle$ and $\mathbf{b} = \langle 3, 1, 2 \rangle$. Find

(a) $|\mathbf{b} - \mathbf{a}|$.

(b) a unit vector that has the same direction as \mathbf{b} .

Dot Product of two nonzero vectors \mathbf{a} and \mathbf{b} is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta,$$

where θ is the angle between \mathbf{a} and \mathbf{b} , $0 \leq \theta \leq \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If θ is the *angle* between two nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \underline{\hspace{10cm}}$$

DEFINITION 5. Two nonzero vectors \mathbf{a} and \mathbf{b} are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$.

EXAMPLE 6. For two nonzero vectors \mathbf{a} and \mathbf{b} prove that

(a)

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

(b)

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

EXAMPLE 7. For what value(s) of c are the vectors $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ orthogonal?

EXAMPLE 8. The points $A(6, -1, 0)$, $B(-3, 1, 2)$, $C(2, 4, 5)$ form a triangle. Find angle at A .

DEFINITION 9. The **work** done by a force \mathbf{F} in moving an object from point A to point B is given by

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ is the distance the object has moved (or displacement).

EXAMPLE 10. A force is given by a vector $\mathbf{F} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(1, 2, 0)$ to the point $Q(2, 3, 5)$. Find the work done.