# F18LN_12_2_and_12_3

Monday, August 26, 2019 1:19 AM



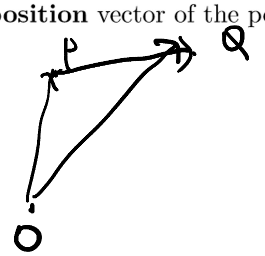
## 12.2 & 12.3: Vectors and the Dot Product

DEFINITION 1. *A* 3**-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0,0,0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point $P$.




$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1,2,3)$ *and terminal point* $B(3,2,-1)$*. What is the position vector of the point* $A$*?*

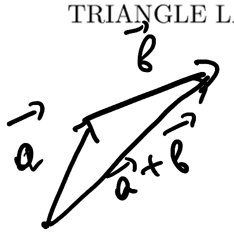
$\overrightarrow{AB} = \langle 3-1, 2-2, -1-3 \rangle = \langle 2, 0, -4 \rangle$

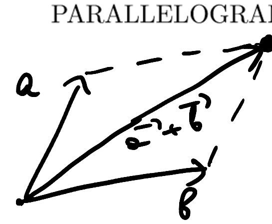
$\overrightarrow{OA} = \langle 1, 2, 3 \rangle$

**Vector Arithmetic:** Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha \mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW PARALLELOGRAM LAW




Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha \mathbf{a}$. Equivalently:

$\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow \quad \vec{b} = \lambda \vec{a}$

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Rewrite in coordinates $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$(\pm) \quad \begin{cases} b_1 = \lambda a_1 \\ b_2 = \lambda a_2 \\ b_3 = \lambda a_3 \end{cases}$$

if all $a_1, a_2, a_3$ are not zero $\Rightarrow \dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3}$



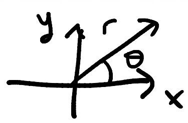
**The magnitude or length** of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector in the same direction as a:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vector* **a** *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector **a** can be determined by its length and the angle from the positive $x$-axis:




In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},$$

i.e. **a** is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2, 5, 0)$ *and* $N(0, 0, 4)$.

$\vec{a} \| \vec{MN}$, $|\vec{a}| = \sqrt{5}$

$$\vec{a} = \pm\sqrt{5}\frac{\vec{MN}}{|\vec{MN}|} = \pm\sqrt{5}\frac{\langle -2, -5, 4 \rangle}{\sqrt{4+25+16}} = \pm\sqrt{5}\frac{\langle -2, -5, 4 \rangle}{\sqrt{45}} = \pm\frac{\langle -2, -5, 4 \rangle}{3} = \pm\left\langle -\frac{2}{3}, -\frac{5}{3}, \frac{4}{3} \right\rangle$$

$9 \cdot 5 \Rightarrow \sqrt{45} = 3\sqrt{5}$

the unit vector in direction of $\vec{MN}$

**Standard Basis Vectors**:
$\mathbf{i} = \langle 1, 0, 0 \rangle$
$\mathbf{j} = \langle 0, 1, 0 \rangle$
$\mathbf{k} = \langle 0, 0, 1 \rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$a_1\langle 1,0,0\rangle \quad a_2\langle 0,1,0\rangle \quad a_3\langle 0,0,1\rangle$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$



EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3 \rangle$ *and* $\mathbf{b} = \langle 3, 1, 2 \rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}|$.

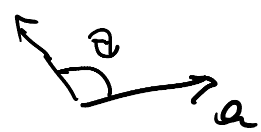
$$\vec{b} - \vec{a} = \langle 3-1, 1-0, 2-(-3) \rangle = \langle 2, 1, 5 \rangle \quad (2+3)$$

$$|\vec{b} - \vec{a}| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{4 + 1 + 25} = \sqrt{30}$$

(b) *a unit vector that has the same direction as* $\mathbf{b} - \mathbf{a}$.

$$\frac{\vec{b} - \vec{a}}{|\vec{b} - \vec{a}|} = \frac{\langle 2, 1, 5 \rangle}{\sqrt{30}} = \left\langle \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle$$

**Dot Product** of two nonzero vectors **a** and **b** is the NUMBER:




$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta,$$

where $\theta$ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If $\theta$ is the *angle* between two nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

DEFINITION 5. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

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EXAMPLE 6. *For two nonzero vectors* **a** *and* **b** *prove that*

**(a)**

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

If $\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \underbrace{\cos\frac{\pi}{2}}_{0} = 0$

If $\vec{a} \cdot \vec{b} = 0 \Rightarrow \underbrace{|\vec{a}|}_{\neq 0} \underbrace{|\vec{b}|}_{\neq 0} \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \vec{a} \perp \vec{b}$



**(b)**

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

$a \cdot a = a_1^2 + a_2^2 + a_3^2$

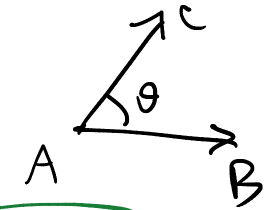
$\sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + a_3^2} = |a|$

EXAMPLE 7. *For what value(s) of* $c$ *are the vectors* $\underbrace{c\mathbf{i} + 2\mathbf{j} + \mathbf{k}}_{\vec{a}}$ *and* $\underbrace{4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}}_{\vec{b}}$ *orthogonal?* (perpendicular)

$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0 \Leftrightarrow 4c + 2 \cdot 3 + 1 \cdot c = 0$

$\Leftrightarrow 5c + 6 = 0 \Leftrightarrow c = \boxed{-\frac{6}{5}}$

EXAMPLE 8. *The points* $A(6,-1,0)$, $B(-3,1,2)$, $C(2,4,5)$ *form a triangle. Find angle at* $A$.




$$\overrightarrow{AB} = \langle -3-6,\ 1-(-1),\ 2-0\rangle = \langle -9, 2, 2\rangle$$

$$\overrightarrow{AC} = \langle 2-6,\ 4-(-1),\ 5-0\rangle = \langle -4, 5, 5\rangle$$

$$\boxed{\cos\theta} = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{(-9)\cdot(-4)+2\cdot 5+2\cdot 5}{\sqrt{(-9)^2+2^2+2^2}\ \sqrt{(-4)^2+5^2+5^2}} =$$

$$= \frac{36+10+10}{\sqrt{81+4+4}\ \sqrt{16+25+25}} = \boxed{\frac{56}{\sqrt{89}\ \sqrt{66}}} \Rightarrow \theta \approx 43^\circ$$



DEFINITION 9. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* $A$ *to point* $B$ *is given by*

$$W = \mathbf{F}\cdot\mathbf{D}$$

*where* $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

EXAMPLE 10. *A force is given by a vector* $\mathbf{F} = \mathbf{i}-\mathbf{j}+5\mathbf{k}$ *and moves a particle from the point* $P(1,2,0)$ *to the point* $Q(2,3,5)$. *Find the work done.*

$\langle 1, -1, 5\rangle$

$$\overrightarrow{PQ} = \langle 2-1,\ 3-2,\ 5-0\rangle = \langle 1, 1, 5\rangle$$

$$W = \overrightarrow{F}\cdot\overrightarrow{PQ} = 1-1+25 = \boxed{25}$$

**Symmetric equations of the line:** If $abc \neq 0$ then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 3. *Find symmetic equations of lines from Example 1.*

**Vector equation of the line:**

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c \rangle$ is some vector which is parallel to the line, $t$ is a parameter, $-\infty < t < \infty$.

EXAMPLE 4. *Find vector equation of the line that passes through the points $P(1, 1, -4)$ and $Q(0, 3, -4)$.*

EXAMPLE 5. *Determine whether the lines*

$$L_1: \quad x-1=\frac{y+2}{3}=\frac{z-4}{-1}$$

*and*

$$L_2: \quad x=2t, \quad y=3+t, \quad z=-3+4t$$

*are parallel, skew, or intersecting.*

**Summarizing table**

| Vector equation | Parametric equations | Symmetric equations |
|---|---|---|
| $\mathbf{r}(t)=\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle,$ <br> $-\infty<t<\infty$ | $x = x_0+at,$ <br> $y = y_0+bt,$ <br> $z = z_0+ct,$ <br> $-\infty<t<\infty$ | If $abc\neq 0$ then <br> $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ <br> If, for example, $a=0$ then the symmetric equations have the form: <br> $x=x_0, \quad \frac{y-y_0}{b}=\frac{z-z_0}{c}$ |

## Line segments

*How to find parametric equation of a line segment:*

1. *Find parametric equation for the entire line;*

2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 6. *Find parametric equations describing the line segment joining the points* $M(1, 2, 3)$ *and* $N(3, 2, 1)$.

# Planes

## Planes parallel to the coordinate planes:

## Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for $P_0$ and $P$ respectively.

**Vector equation of the plane:** $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}.$

**Scalar equation of plane:**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Often this will be written as a **linear equation** in $x, y, z$,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 7. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4 \rangle$*. Find the intercepts and sketch the plane.*

EXAMPLE 8. *Determine the equation of the plane through the points* $A(1, 1, 1)$*,* $B(0, 1, 0)$ *and* $C(1, 2, 3)$.

Two planes are **parallel** if their normal vectors are parallel.

Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 9. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

*Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.*

**(a)** $P_1$ *and* $P_2$

**(b)** $P_1$ *and* $P_3$

**(c)** $P_2$ *and* $P_3$

**(d)** $P_1$ *and* $P_4$

**Line as an intersection of two non parallel planes**:

$$L : \left\{ \begin{array}{ccccccccc} a_1x & + & b_1y & + & c_1z & + & d_1 & = & 0 \\ a_2x & + & b_2y & + & c_2z & + & d_2 & = & 0 \end{array} \right.$$

The direction vector of $L$ is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2$.

EXAMPLE 10. *Find an equation of the line given as intersection of two planes:*

$$\begin{array}{ccccccc} x & - & y & + & 3z & = & 0 \\ x & + & y & + & 4z & = & 2 \end{array}$$