F18LN_12_2_and_12_3

Monday, August 26, 2019 1:19 AM

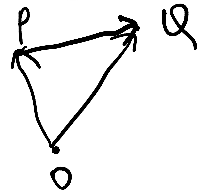
12.2 & 12.3: Vectors and the Dot Product

DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1, 2, 3)$ *and terminal point* $B(3, 2, -1)$. *What is the position vector of the point* A*?*

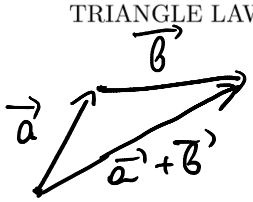
$$\overrightarrow{AB} = \langle 3-1, 2-2, -1-3 \rangle = \langle 2, 0, -4 \rangle$$

$$\overrightarrow{OA} = \langle 1, 2, 3 \rangle$$

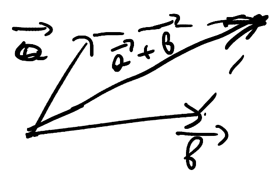
Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW



PARALLELOGRAM LAW



Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \Leftrightarrow \vec{b} = \alpha \vec{a}$$

Rewrite in coordinates: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{b} = \alpha \vec{a} \Leftrightarrow \langle b_1, b_2, b_3 \rangle = \alpha \langle a_1, a_2, a_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle \Leftrightarrow$$

$$\begin{cases} b_1 = \alpha a_1 \\ b_2 = \alpha a_2 \\ b_3 = \alpha a_3 \end{cases}$$

If all a_1, a_2, a_3 are not zero then

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$$

The magnitude or length of $a = \langle a_1, a_2, a_3\rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

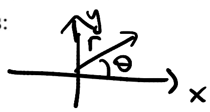
$$|\alpha \vec{a}| = |\alpha|\,|\vec{a}|$$

`Zero vector:` $\mathbf{0} = \langle 0, 0, 0\rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

$\vec{a} \neq \vec{0}$

`Unit vector in the same direction as a:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vectoe* **a** *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector **a** can be determined by its length and the angle from the positive x-axis:



In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},$$

i.e. **a** is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2, 5, 0)$ *and* $N(0, 0, 4)$.

$\vec{a} \parallel \overrightarrow{MN}$

$|\vec{a}| = \sqrt{5}$

$$\vec{a} = \pm\sqrt{5}\,\frac{\overrightarrow{MN}}{|\overrightarrow{MN}|} = \pm\sqrt{5}\,\frac{\langle -2, -5, 4\rangle}{\sqrt{4+25+16}} = \pm\sqrt{5}\,\frac{\langle 2, -5, 4\rangle}{\sqrt{45}} = \pm\cancel{\sqrt{5}}\,\frac{\langle -2, -5, 4\rangle}{3\cancel{\sqrt{5}}} = \pm\left\langle -\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}\right\rangle$$

$9\cdot 5 \Rightarrow \sqrt{45} = 3\sqrt{5}$

Standard Basis Vectors:
$\mathbf{i} = \langle 1, 0, 0\rangle$
$\mathbf{j} = \langle 0, 1, 0\rangle$
$\mathbf{k} = \langle 0, 0, 1\rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
We have:

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = \underbrace{\langle a_1, 0, 0\rangle} + \underbrace{\langle 0, a_2, 0\rangle} + \underbrace{\langle 0, 0, a_3\rangle}$$

$$a_1\underbrace{\langle 1, 0, 0\rangle}_{\hat{i}} + a_2\underbrace{\langle 0, 1, 0\rangle}_{\hat{j}} + a_3\underbrace{\langle 0, 0, 1\rangle}_{\hat{k}} =$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3\rangle$ *and* $\mathbf{b} = \langle 3, 1, 2\rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}|$.

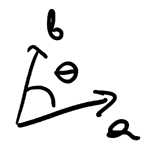
$$\vec b - \vec a = \langle 3-1,\ 1-0,\ \underbrace{2-(-3)}_{2+3}\rangle = \langle 2, 1, 5\rangle$$

$$|\vec b - \vec a| = \sqrt{2^2+1^2+5^2} = \boxed{\sqrt{30}}$$

(b) *a unit vector that has the same direction as* $\vec b - \vec a$

$$\frac{\vec b - \vec a}{|\vec b - \vec a|} = \frac{\langle 2, 1, 5\rangle}{\sqrt{30}} = \left\langle \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle$$

Dot Product of two nonzero vectors **a** and **b** is the NUMBER:



$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta,$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If θ is the *angle* between two nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{a_1b_1 - a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\ \sqrt{b_1^2+b_2^2+b_3^2}}$$

DEFINITION 5. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

EXAMPLE 6. *For two nonzero vectors* **a** *and* **b** *prove that*

(a)

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

If $\vec a \perp \vec b$, then $\theta = \frac{\pi}{2} \Rightarrow \vec a \cdot \vec b = |\vec a|\,|\vec b| \underbrace{\cos\frac{\pi}{2}} = 0$

If $\vec a \cdot \vec b = 0 \Rightarrow \underbrace{|\vec a|}_{\neq 0}\,\underbrace{|\vec b|}_{\neq 0} \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow a \perp b$

(b)

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 \Rightarrow \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} = |\vec{a}|$

→ perpendicular

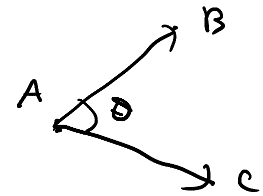
EXAMPLE 7. *For what value(s) of c are the vectors $\underbrace{c\mathbf{i} + 2\mathbf{j} + \mathbf{k}}_{\vec{a}}$ and $\underbrace{4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}}_{\vec{b}}$ orthogonal?*

$\vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0$

$\vec{a}\cdot\vec{b} = 4c + \underbrace{2\cdot 3}_{6} + 1\cdot c = 0 \Rightarrow 5c + 6 = 0 \Rightarrow \boxed{c = -\frac{6}{5}}$

EXAMPLE 8. *The points $A(6,-1,0)$, $B(-3,1,2)$, $C(2,4,5)$ form a triangle. Find angle at A.*

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|}$$



$\vec{AB} = \langle -3-6, 1-(-1), 2-0 \rangle = \langle -9, 2, 2 \rangle \Rightarrow$

$|\vec{AB}| = \sqrt{81+4+4} = \sqrt{89}$

$\vec{AC} = \langle 2-6, 4-(-1), 5-0 \rangle = \langle -4, 5, 5 \rangle \rightarrow |\vec{AC}| = \sqrt{16+25+25} = \sqrt{66}$

$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{36+10+10}{\sqrt{89}\sqrt{66}} = \frac{56}{\sqrt{89}\sqrt{66}}$ $(\Rightarrow \theta \approx 43^\circ)$

DEFINITION 9. *The* **work** *done by a force* $\mathbf{F}$ *in moving an~~d~~ object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

EXAMPLE 10. *A force is given by a vector* $\mathbf{F} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ *and moves a particle from the point* $P(1,2,0)$ *to the point* $Q(2,3,5)$. *Find the work done.*

$= \langle 1, -1, 5 \rangle$

$$W = F \cdot \overrightarrow{PQ} = \langle 1, -1, 5 \rangle \cdot \underbrace{\langle 2-1, 3-2, 5-0 \rangle}_{\langle 1, 1, 5 \rangle} =$$

$$= 1 \cdot 1 + 1 \cdot (-1) + 5 \cdot 5 = 1 - 1 + 25 = \boxed{25}$$