F19_LN_12_5_completed_501

Sunday, September 8, 2019 8:32 PM

F19_LN_1...

F19_LN_12_5

Monday, September 2, 2019 7:48 AM

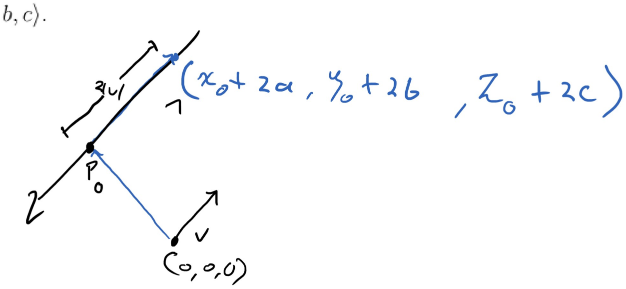
F19_LN_12_5

12.5: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$.



Parametric equations of the line:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

EXAMPLE 1. *Find parametric equations of the line*

(a) *passing through the point* $(3, -4, 1)$ *and parallel to* $\mathbf{v} = \langle 7, 0, -1 \rangle$

$x = 3 + t7$
$y = -4 + t \cdot 0 = -4$
$z = 1 - t.$

(b) *passing through the origin and parallel to* $\mathbf{v} = \langle 5, 5, 5 \rangle$

$x = 0 + 5t = 5t$
$y = 5t$
$z = 5t$

EXAMPLE 2. *Consider the line L that passes through the points $A(1, 1, 1)$ and $B(2, 3, -2)$. Find points at that L intersects the yz-plane.*

$\overrightarrow{AB} = \langle 2-1, 3-1, -2-1 \rangle = \langle 1, 2, -3 \rangle$

$x = 1 + 1t = 1 + t$
$y = 1 + 2t$
$z = 1 + (-3)t = 1 - 3t$

L intersects the yz plane at points for which the x-component is zero, i.e., $0 = x = 1 + t$, so $t = -1$.

when $t=0$ $(x, y, z) = (0, -1, 4)$.

Symmetric equations of the line: If $abc \neq 0$ then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 3. *Find symmetic equations of lines from Example 1.*

a) The line through $(x_0, y_0, z_0) = (3, -4, 1)$ parallel to the vector $\langle a, b, c\rangle = \langle 7, 0, -1\rangle$ is defined by

$$\boxed{\frac{x-3}{7} = \frac{z-1}{-1}, \quad y = -4}$$

b) The line passing through $(0,0,0)$ parallel to $\langle 5,5,5\rangle$ is defined by $\frac{x}{5} = \frac{y}{5} = \frac{z}{5} \iff \boxed{x = y = z}$

Vector equation of the line:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c\rangle$ is some vector which is parallel to the line, t is a parameter, $-\infty < t < \infty$.

or, equivalently, $t \in \mathbb{R}$

EXAMPLE 4. *Find vector equation of the line that passes through the points $P(1, 1, -4)$ and $Q(0, 3, -4)$.*

The displacement vector $\overrightarrow{PQ} = \langle 0-1, 3-1, -4-(-4)\rangle = \langle -1, 2, 0\rangle$ is parallel to the line.

$$\boxed{r(t) = \langle 1, 1, -4\rangle + t\langle -1, 2, 0\rangle}$$

EXAMPLE 5. *Determine whether the lines*

$$L_1: \quad x - 1 = \frac{y+2}{3} = \frac{z-4}{-1}$$

Given by Symmetric equations

and

$$L_2: \quad x = 2t, \quad y = 3 + t, \quad z = -3 + 4t$$

Given by parametric equations

are parallel, skew, or intersecting.

not necessary

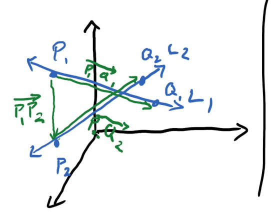
Let's find vector parallel to each line by first finding 2 points on each line.

- Points on L_1: Solving $0 = x - 1 = \frac{y+2}{3} = \frac{z-4}{-1}$ and $1 = x - 1 = \frac{y+2}{3} = \frac{z-4}{-1}$ for (x, y, z), we get points on L_1
$$P_1 = (1, -2, 4) \text{ and } Q_1 = (2, 1, 3)$$
- Points on L_2: Choosing $t = 0$ and $t = 1$, we get points on L_2
$$P_2 = (0, 3, -3) \text{ and } Q_2 = (2, 4, 1).$$

$\overrightarrow{P_1Q_1} = \langle 1, 3, -1 \rangle$ and $\overrightarrow{P_2Q_2} = \langle 2, 1, 4 \rangle$ are parallel to L_1 and L_2, respectivel.

$\overrightarrow{P_1Q_1}$ is not a multiple of $\overrightarrow{P_2Q_2}$

$\boxed{\text{so } L_1 \text{ and } L_2 \text{ are NOT parallel}}$



$\overrightarrow{P_1P_2} = \langle -1, 5, -7 \rangle$

$$\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\hat{i} - 6\hat{j} - 5\hat{k}$$

$$\overrightarrow{P_1P_2} \cdot (\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) = (-\hat{i} + 5\hat{j} - 7\hat{k}) \cdot (13\hat{i} - 6\hat{j} - 5\hat{k}) = -13 - 30 + 35 \neq 0.$$

Therefore $\overrightarrow{P_1Q_1}$, $\overrightarrow{P_2Q_2}$, and $\overrightarrow{P_1P_2}$ are not coplanar, so they $\boxed{L_1 \text{ and } L_2 \text{ are skew}}$

Summarizing table

Vector equation	Parametric equations	Symmetric equations

$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$,
 $-\infty < t < \infty$
	$x = x_0 + at,$
	 $y = y_0 + bt,$
	 $z = z_0 + ct,$
	 $-\infty < t < \infty$
		If $abc \neq 0$ then
		 $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
		 If, for example, $a = 0$ then
		 the symmetric equations have the form:
		 $x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*
2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 6. *Find parametric equations describing the line segment joining the points* $M(1,2,3)$ *and* $N(3,2,1)$.

The line segment is parallel to $\overrightarrow{MN} = \langle 2, 0, -2 \rangle$, the the segment is on the line described by

$$x = 1 + 2t$$
$$y = 2$$
$$z = 3 - 2t$$

We need to find values of t so that (x, y, z) is in the line segment.

When $t=0$, $M = (x, y, z)$. Increasing t moves the point (x, y, z) along the line in the direction from M to N, so we need to find t such that $N = (x, y, z)$, that is, we need $3 = x = 1 + 2t$. Therefore when $t = 1$, $N = (x, y, z)$. The line segment is described by

$$\boxed{\begin{aligned} x &= 1+2t \\ y &= 2 \\ z &= 3-2t \\ 0 &\le t \le 1 \end{aligned}}$$

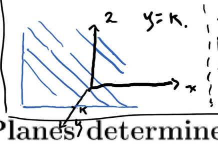
Planes

Planes parallel to the coordinate planes:

A plane parallel to xy-plane is defined by the equation $z = k$ for some constant k.

" " " " xz-plane " " " " " $y = k$ " " " " "

" " " " yz-plane " " " " " $x = k$ " " " " "

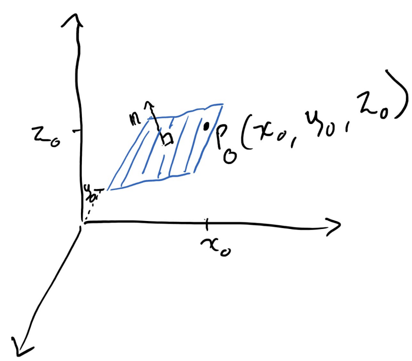


A plane parallel to the xz-plane.

Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.



Vector equation of the plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$

$n \perp r - r_0$

Scalar equation of plane:

$$\underbrace{a(x - x_0) + b(y - y_0) + c(z - z_0)}_{= ax + by + cz - (ax_0 + by_0 + cz_0)} = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

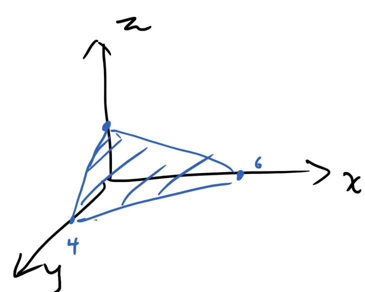
EXAMPLE 7. *Determine the equation of the plane through the point $(1, 2, 1)$ and orthogonal to vector $\langle 2, 3, 4\rangle$. Find the intercepts and sketch the plane.*

$$2(x-1) + 3(y-2) + 4(z-1) = 0$$

Find x-intercept by setting $y = z = 0$: $y=z=0$ implies $2x - 2 - 6 - 4 \iff \boxed{x = 6}$.

y-intercept: $x=z=0$ implies $-2 + 3y - 6 - 4 = 0 \iff \boxed{y = 4}$

z-int.: $x=y=0$ implies $-2 - 6 + 4z - 4 = 0 \iff \boxed{z = 3}$



EXAMPLE 8. *Determine the equation of the plane through the points $A(1, 1, 1)$, $B(0, 1, 0)$ and $C(1, 2, 3)$.*

$\vec{AB} = \langle -1, 0, -1\rangle$, $\vec{AC} = \langle 0, 1, 2\rangle$, and

$$n = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \hat{i} + 2\hat{j} - \hat{k}$$

n is orthogonal to the plane containg $(1,1,1)$, so the plane is described by

$$x - 1 + 2(y-1) - z + 1 = 0$$

$$\boxed{x + 2y - z = 2}$$

Two planes are **parallel** if their normal vectors are parallel.

Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 9. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & 1z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

$\vec{n}_1 = \langle 2, 3, 1 \rangle$
$\vec{n}_2 = \langle -4, -6, -2 \rangle$
$\vec{n}_3 = \langle 2, 0, -4 \rangle$
$\vec{n}_4 = \langle -2, 3, 1 \rangle$

Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.

(a) P_1 *and* P_2

Way 1 $\quad \vec{n}_2 = -2\vec{n}_1 \Rightarrow \vec{n}_1 \parallel \vec{n}_2 \Rightarrow P_1 \parallel P_2$

Way 2 $\quad \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ -4 & -6 & -2 \end{vmatrix} = \vec{0} \Rightarrow \vec{n}_1 \parallel \vec{n}_2$

(b) P_1 *and* P_3

$\vec{n}_1 \cdot \vec{n}_3 = 2\cdot 2 + 3\cdot 0 + 1\cdot(-4) = 4 - 4 = 0$

$\vec{n}_1 \perp \vec{n}_3 \Rightarrow P_1 \perp P_3 \rightarrow$ orthogonal.

(c) P_2 *and* P_3 Since $P_1 \parallel P_2$ and $P_1 \perp P_3 \Rightarrow P_2 \perp P_3$

(d) P_1 *and* P_4

$\vec{n}_1 \cdot \vec{n}_4 = 2\cdot(-2) + 3\cdot 3 + 1\cdot 1 = 6$

$|\vec{n}_1| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$|\vec{n}_2| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \dfrac{6}{14} = \dfrac{3}{7} \Rightarrow \theta = \arccos\dfrac{3}{7} \approx 65^\circ.$

$\Downarrow$

neither parallel nor orthogonal, has angle 65°.

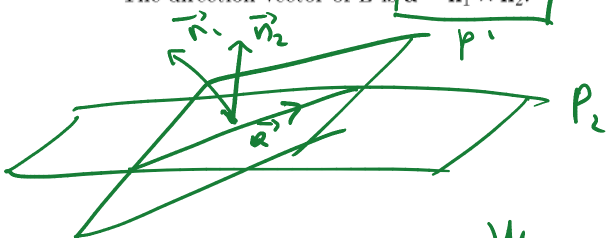
Line as an intersection of two non parallel planes:

$$L : \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 & p^1 \text{ with normal } \vec{n}_1 \\ a_2x + b_2y + c_2z + d_2 = 0 & p^2 \text{ with normal } \vec{n}_2 \end{cases}$$

$\vec{n}_1 = \langle a_1, b_1, c_1 \rangle$

$\vec{n}_2 = \langle a_2, b_2, c_2 \rangle$

The direction vector of L is $\boxed{\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2}$.



$\vec{a} \perp \vec{n}_1$

$\vec{a} \perp \vec{n}_2$

$\Downarrow$

$\vec{a} \parallel \vec{n}_1 \times \vec{n}_2$

We can take $\vec{a} = \vec{n}_1 \times \vec{n}_2$

EXAMPLE 10. *Find an equation of the line given as intersection of two planes:*

$$\begin{aligned} x - y + 3z &= 0 \\ x + y + 4z &= 2 \end{aligned}$$

$\vec{n}_1 = \langle 1, -1, 3 \rangle$

$\vec{n}_2 = \langle 1, 1, 4 \rangle$

The desired lines is parallel to $\vec{a} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} =$

$= \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \hat{k} =$

$= -7\hat{i} - \hat{j} + 2\hat{k} = \langle -7, -1, 2 \rangle$

Find one point on the line by setting $z = 0$ and find x & y

$\begin{cases} x - y = 0 \\ x + y = 2 \end{cases} \Rightarrow$ Eliminate y by adding $2x = 2 \Rightarrow x = 1 \Rightarrow y = x = 1 \Rightarrow (1,1,0)$ is on the line

The parametric equation of the line:

$$\begin{cases} x = 1 - 7t \\ y = 1 - t \\ z = 2t \end{cases}$$